A weak reflection compatible with tail club guessing via semiproper iteration(The interplay between set theory of the reals and iterated forcing)

MIYAMOTO, Tadatoshi

数理解析研究所講究録
1530 巻 2007 年 49-81

A weak reflection compatible with tail club guessing via semiproper iteration

Tadatoshi MIYAMOTO 宮元 忠敏

10th, January, 2007

Mathematics, Nanzan University 南山大学 数理情報

27 Seirei-cho, Seto-shi 489-0863 Japan

miyamoto@nanzan-u.ac.jp

Abstract

We formulate a weak form of reflection principle. This principle is compatible with tail club guessing on the first uncountable cardinal and implies the corresponding tail club guessing ideal is saturated. We consider a forcing axiom which implies our principle and is compatible with tail club guessing via a semiproper iterated forcing.

Introduction

In [I], a model of set theory is constructed where a tail club guessing ideal is saturated. It also shows consistencies of forcing axioms compatible with tail club guessing. On the other hand, it is known that many consequences of the Martin's Maximum (MM) are gotten interpolated by the Strong Reflection Principle (SRP) in [B]. For example, SRP implies the nonstationary ideal NS_{ω_1} is saturated. We intend to do the same explicitly in the context of [I].

To formulate a suitable principle which looks like SRP, we prefix a ladder system $\langle C_\delta \mid \delta \in A\rangle$ on a subset A of ω_1 as a parameter once for all. If the parameter is tail club guessing and there exists a supercompact cardinal, then we can force a forcing axiom while preserving this tail club guessing. This forcing axiom implies a SRP-like principle which in turn implies the corresponding tail club guessing ideal is saturated. Furthermore, the plus-type forcing axiom for a σ-closed p.o. set together with this tail club guessing on A negates the saturation of NS_{ω_1}. This observation is a modification of [I].

The relevant class of preorders are σ-Baire. They are proper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper. We note that ω-semiproper preorders are $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper. But $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper preorders may not be semiproper. We provide a characterization of preorders which are $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper in terms of preserving a type of stationary sets. This follows some of what [S] considers. We show iteration theorem for semiproper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper under a type of revised countable support iterated forcing found in [M]. More specifically, this note contains the following.

§1. Statements equivalent to SRP are considered to motivate §4.

§2. Two technical lemmas are recorded for §1 and §6.

§3. Notations fixed for tail club guessing ideals.

§4. (proper, $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, plain)-reflection principle introduced.

§5. A tail club guessing ideal can be saturated under the principle in §4.

§6. A proper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper p.o. set is considered to force the principle in §4.

§7. The nonstationary ideal on ω_1 can not be saturated, if MA^+(a σ-closed p.o. set) and a tail club guessing hold.

§8-§12. Trees of clubs and ω-stationary sets etc are introduced and their basic properties are recorded.

§13. $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper p.o. sets get characterized in terms of preserving a type of semistationary sets of §8-§12.

§14-§18. Iteration lemma for semiproper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper and corresponding forcing axiom are considered.

§5 and §7 are modifications of [I]. To go through §14-§18, we used iteration lemma for semiproper from [M]. The principle introduced here can be pushed a bit stronger. It implies a form of the Chang's Conjecture.

Its account may appear elsewhere. I would like to thank members of set theory seminar at Nagoya University for pointing out some simplifications.

§1. Another look at the Strong Reflection Principle

We begin by fixing notations.

1.1 Definition. Let θ be a regular cardinal, then $N \prec H_\theta$ means that $(N, \in)$ is a countable elementary substructure of $(H_\theta, \in)$. For $N, M \prec H_\theta$, $N \subseteq_{\omega_1} M$ means that $N \subseteq M$ and $N \cap \omega_1 = M \cap \omega_1$. Let $\xi \leq \omega_1$. We say $\langle N_\alpha \mid \alpha < \xi \rangle$ is an *$\in$-chain in H_θ*, if

- For all $\alpha < \xi$, $N_\alpha \prec H_\theta$,
- For all $\beta < \xi$ with $\beta + 1 < \xi$, $\langle N_\alpha \mid \alpha \leq \beta \rangle \in N_{\beta+1}$,
- For all limit $\beta < \xi$, $N_\beta = \bigcup\{N_\alpha \mid \alpha < \beta\}$.

We are interested in cases when $\xi = \omega_1, \omega + 1$ and ω.

For two $\in$-chains $\langle N_n \mid n < \omega \rangle, \langle M_n \mid n < \omega \rangle$ in H_θ, the notation

$$\langle N_n \mid n < \omega \rangle \subseteq_{\omega_1} \langle M_n \mid n < \omega \rangle$$

means for all $n < \omega$, $N_n \subseteq_{\omega_1} M_n$.

The following modifies [B] a little.

1.2 Definition. We say *Strong Reflection Principle (SRP)* holds, if for any (K, S, θ, a) such that

- $K \supseteq \omega_1$,
- $S \subseteq [K]^\omega$,
- θ is a regular cardinal with $K \in H_{|\mathrm{TC}(K)|^+} \in H_{(2^{|\mathrm{TC}(K)|})^+} \in H_\theta$,
- $a \in H_\theta$,

there exists $(\langle N_\alpha \mid \alpha < \omega_1 \rangle, C)$ such that

- $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ is an $\in$-chain in H_θ,
- $a \in N_0$,
- $C \subseteq \omega_1$ is a club,
- For each $\alpha \in C$, we have either (1) or (2),

(1) $N_\alpha \cap K \in S$.

(2) For any N with $N_\alpha \subseteq_{\omega_1} N \prec H_\theta$, we have $N \cap K \notin S$.

The following is from [F].

1.3 Definition. Let $K \supseteq \omega_1$. For $S \subseteq [K]^\omega$, S is *projectively stationary*, if for any stationary $E \subseteq \omega_1$, we have $\{a \in S \mid a \cap \omega_1 \in E\}$ is stationary in $[K]^\omega$.

We recap [F] as follows to motivate our principle.

1.4 Proposition. *The following are equivalent.*

(1) SRP holds.

(2) For any (K, S, θ, a) as in SRP, there exists an $\in$-chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ and a club $C \subseteq \omega_1$ such that for each $\alpha \in C$, we have either (2.1) or (2.2),

(2.1) $N_\alpha \cap K \in S$.

(2.2) For any $(\langle \alpha_n \mid n < \omega \rangle, \langle N'_n \mid n \leq \omega \rangle)$ such that

- *$\alpha_n < \alpha$ are strictly increasing and $\sup\{\alpha_n \mid n < \omega\} = \alpha$,*

- $\langle N'_n \mid n \leq \omega \rangle$ *is an* $\in$*-chain in* H_θ,
- $\langle N_{\alpha_n} \mid n < \omega \rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega \rangle$,

we have $N'_\omega \cap K \notin S$.

(3) For any (K, S, θ, a) *as in SRP except* S *is assumed to be projectively stationary in* $[K]^\omega$*, there exists an* $\in$*-chain* $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ *and a club* $C \subseteq \omega_1$ *such that for each* $\alpha \in C$*, we have* $N_\alpha \cap K \in S$.

Proof. (1) implies (2): Let (K, S, θ, a) be given as in (2). Then by (1), we have $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ and C. We claim these two work in (2). To see this, let $\alpha \in C$ and suppose we are in case (2) of (1). If $\langle N_{\alpha_n} \mid n < \omega \rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega \rangle$, then $N_\alpha \subseteq_{\omega_1} N'_\omega$ and so $N'_\omega \cap K \notin S$.

(2) implies (3): Let (K, S, θ, a) be as in (3). So in particular, S is assumed to be projectively stationary in $[K]^\omega$. By (2), we have $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ and C. Then this $\in$-chain in H_θ itself and the following club work.

Claim 1. $\{\alpha \in C \mid N_\alpha \cap K \in S\}$ *contains a club.*

Proof. By contradiction. Suppose otherwise and let $E = \{\alpha \in C \mid \text{(2.2) holds at } \alpha\}$. Then E is stationary. Hence we may take a sufficiently large regular cardinal λ and $M \prec H_\lambda$ such that

- $\langle N_\alpha \mid \alpha < \omega_1 \rangle \in M$,
- $M \cap K \in S$,
- $M \cap \omega_1 = \delta \in E$.

Since $H_\theta \in H_\lambda$, we may assume there exists an $\in$-chain $\langle N'_n \mid n \leq \omega \rangle$ in H_θ with $N'_\omega = M \cap H_\theta$. Let $\delta_n = N'_n \cap \omega_1$. Then δ_n are strictly increasing and $\sup\{\delta_n \mid n < \omega\} = \delta$. Since we may assume $\langle N_\alpha \mid \alpha < \omega_1 \rangle \in N'_n$, we have $N_{\delta_n} \subseteq_{\omega_1} N'_n$ and so $\langle N_{\delta_n} \mid n < \omega \rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega \rangle$. Hence $M \cap K = N'_\omega \cap K \notin S$. This is a contradiction.

(3) implies (1): Let (K, S, θ, a) be as in (1). Let $\overline{N} \in S^\perp$, if

- $\overline{N} \prec H_{(2^{|\mathrm{TC}(K)|})^+}$,
- Either the following (1) or (2) holds,

  (1) $\overline{N} \cap K \in S$.

  (2) For any $\overline{M} \prec H_{(2^{|\mathrm{TC}(K)|})^+}$ with $\overline{N} \subseteq_{\omega_1} \overline{M}$, we have $\overline{M} \cap K \notin S$.

Claim 2. $S^\perp \subset [H_{(2^{|\mathrm{TC}(K)|})^+}]^\omega$ *is projectively stationary.*

Proof. Let E be any stationary subset of ω_1 and let

$$\phi : [H_{(2^{|\mathrm{TC}(K)|})^+}]^{<\omega} \longrightarrow H_{(2^{|\mathrm{TC}(K)|})^+}.$$

We want $\overline{N} \in S^\perp$ such that $\overline{N} \cap \omega_1 \in E$ and $\overline{N}$ is ϕ-closed. To this end, take $M \prec H_\theta$ such that $\phi \in M$ and $M \cap \omega_1 \in E$. We argue in two cases.

Case 1. For all $M' \prec H_\theta$ with $M \subseteq_{\omega_1} M'$, we have $M' \cap K \notin S$: In this case we

Claim 3. *For all* $\overline{M} \prec H_{(2^{|\mathrm{TC}(K)|})^+}$ *with* $M \cap H_{(2^{|\mathrm{TC}(K)|})^+} \subseteq_{\omega_1} \overline{M}$*, we have* $\overline{M} \cap K \notin S$.

Proof. This is because $H_{|\mathrm{TC}(K)|^+} \in H_{(2^{|\mathrm{TC}(K)|})^+} \in H_\theta$. More precisely, given $\overline{M}$, by 2.1 lemma (three H lemma) of next section, we have $M' \prec H_\theta$ such that $M \subseteq M'$ and $\overline{M} \cap H_{|\mathrm{TC}(K)|^+} = M' \cap H_{|\mathrm{TC}(K)|^+}$. Hence $M \cap \omega_1 = \overline{M} \cap \omega_1 = M' \cap \omega_1$ and $\overline{M} \cap K = M' \cap K \notin S$. Hence $\overline{M} \cap K \notin S$.

□

Let $\overline{N} = M \cap H_{(2^{|\mathrm{TC}(K)|})^+}$. Then $\overline{N} \in S^\perp$, $\overline{N}$ is ϕ-closed and $\overline{N} \cap \omega_1 = M \cap \omega_1 \in E$.

Case 2. There exists $M' \prec H_\theta$ such that $M \subseteq_{\omega_1} M'$ and $M' \cap K \in S$: Let $\overline{N} = M' \cap H_{(2^{|\mathrm{TC}(K)|})^+}$. Then this $\overline{N}$ works.

□

Now apply (3) with $\left(H_{(2^{|\mathrm{TC}(K)|})^+}, S^\perp, \lambda, (a, H_{(2^{|\mathrm{TC}(K)|})^+}, H_\theta)\right)$. There exists $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ and C such that for all $\alpha \in C$, we have $M_\alpha \cap H_{(2^{|\mathrm{TC}(K)|})^+} \in S^\perp$. Since $H_\theta \in M_0$, $\langle M_\alpha \cap H_\theta \mid \alpha < \omega_1 \rangle$ is an $\in$-chain in H_θ. We claim this $\in$-chain and C work. To see this let $\alpha \in C$. If $M_\alpha \cap K \in S$, then we have $(M_\alpha \cap H_\theta) \cap K = M_\alpha \cap K \in S$. Otherwise for any $M \prec H_\theta$ with $M_\alpha \cap H_\theta \subseteq_{\omega_1} M$, we have $M_\alpha \cap H_{(2^{|\mathrm{TC}(K)|})^+} \subseteq_{\omega_1} M \cap H_{(2^{|\mathrm{TC}(K)|})^+}$ and so $M \cap K \notin S$.

□

§2. Technical Lemma

We frequently make use of the following crutial technical lemma from [B].

2.1 Lemma. (Three H lemma) *Let $\omega_1 \leq \kappa < \theta < \lambda$ be regular cardinals with $H_\kappa \in H_\theta \in H_\lambda$. Let $N \prec H_\lambda$ with $\kappa, \theta \in N$. Then we have*

(1) $H_\kappa, H_\theta \in N$, $N \cap H_\kappa \prec H_\kappa$ and $N \cap H_\theta \prec H_\theta$.

Let $\overline{M} \prec H_\theta$ with $N \cap H_\theta \subseteq_{\omega_1} \overline{M}$. Then

(2) If $M = \{f(s) \mid f \in N,\ s \in \overline{M} \cap H_\kappa\}$, then

- *$N \subseteq_{\omega_1} M \prec H_\lambda$,*
- *$M \cap H_\kappa = \overline{M} \cap H_\kappa$.*

Proof. For (1): Since $H_\kappa, H_\theta \in H_\lambda$, we may check H_κ and H_θ are definable in H_λ from κ and θ, respectively. Hence $H_\kappa, H_\theta \in N \prec H_\lambda$. Since $H_\kappa, H_\theta \in N \prec H_\lambda$, we may check that $N \cap H_\kappa \prec H_\kappa$ and $N \cap H_\theta \prec H_\theta$.

For (2): Let $f_1(s_1), \cdots, f_n(s_n) \in M$ and $\phi(v_0, v_1, \cdots, v_n)$ be a formula. Then there exists $g : H_k \longrightarrow H_\lambda$ such that if $a_1, \cdots, a_n \in H_\kappa$ and $H_\lambda \models$ "$\exists y\ \phi(y, f_1(a_1), \cdots, f_n(a_n))$", then

$$H_\lambda \models \text{“}\phi(g((a_1, \cdots, a_n)), f_1(a_1), \cdots, f_n(a_n))\text{”}.$$

Since $f_1, \cdots, f_n, H_\kappa \in N$, we may assume $g \in N$.

Now if $H_\lambda \models$ "$\exists y\ \phi(y, f_1(s_1), \cdots, f_n(s_n))$" with some $s_1, \cdots, s_n \in \overline{M} \cap H_\kappa$, then we have

$$H_\lambda \models \text{“}\phi(g((s_1, \cdots, s_n)), f_1(s_1), \cdots, f_n(s_n))\text{”}.$$

Since $(s_1, \cdots, s_n) \in \overline{M} \cap H_\kappa$, we have $g((s_1, \cdots, s_n)) \in M$. Hence by Tarski's criterion, we conclude

$$M \prec H_\lambda.$$

To see $N \subseteq M$, let $x \in N$. Then $x = f(\emptyset)$, where $f = \{(a, x) \mid a \in H_\kappa\}$ is the constant function in N. Hence $x \in M$.

To see $\overline{M} \cap H_\kappa \subseteq M$, let $f = \{(a, a) \mid a \in H_\kappa\}$ be the identity function on H_κ. Then $f \in N$ and $f(s) = s$ for all $s \in \overline{M} \cap H_\kappa$. Hence $\overline{M} \cap H_\kappa \subseteq M$.

To see $\overline{M} \cap H_\kappa \supseteq M \cap H_\kappa$, let $f(s) \in M \cap H_\kappa$. We may assume $f \in N$ and $f : H_\kappa \longrightarrow H_\kappa$. Since $f \subset H_\kappa \times H_\kappa$, we have $f \subset H_\kappa \in H_\theta$ and so $f \in N \cap H_\theta \subseteq \overline{M}$. Hence $f(s) \in \overline{M} \cap H_\kappa$.

Since we have $\overline{M} \cap H_\kappa = M \cap H_\kappa$, we have $N \cap \omega_1 = \overline{M} \cap \omega_1 = M \cap \omega_1$ and so $N \subseteq_{\omega_1} M$.

□

2.2 Corollary. (Sequential three H lemma) *Let κ, θ, λ be as above. Let $\langle N_n \mid n < \omega \rangle$ be an $\in$-chain in H_λ with $\kappa, \theta \in N_0$. Let $\langle \overline{M}_n \mid n < \omega \rangle$ be an $\in$-chain in H_θ such that*

$$\langle N_n \cap H_\theta \mid n < \omega \rangle \subseteq_{\omega_1} \langle \overline{M}_n \mid n < \omega \rangle.$$

Then there exists an $\in$-chain $\langle M_n \mid n < \omega \rangle$ in H_λ such that for each $n < \omega$, we have $N_n \subseteq_{\omega_1} M_n$ and $M_n \cap H_\kappa = \overline{M_n} \cap H_\kappa$.

Proof. We just observe $M_n \in M_{n+1}$. But this follows from the fact that

$$M_n = \{f(s) \mid f \in N_n,\ s \in \overline{M}_n \cap H_\kappa\}$$

is definable from N_n and $\overline{M_n} \cap H_\kappa$. But we have

$$N_n \in N_{n+1} \subseteq M_{n+1},$$

$$\overline{M}_n \cap H_\kappa \in \overline{M}_{n+1} \cap H_\kappa \subseteq M_{n+1}.$$

□

§3. Tail club guessing ideals

We recap tail club guessing ideals and fix our notation (see [I] for more on this).

3.1 Definition. We say $\langle C_\delta \mid \delta \in A \rangle$ is a *ladder system on A*, if $A \subseteq \{\delta < \omega_1 \mid \delta \text{ is limit}\}$ and for each $\delta \in A$, C_δ is a cofinal subset of δ and is of order-type ω. When C_δ gets enumerated increasingly, we denote the n-th element either by $C_\delta(n)$ or simply δ_n. For a club $D \subseteq \omega_1$, we write

$$X^*(D) = \{\delta \in A \mid C_\delta \subseteq^* D\}$$

where $C_\delta \subseteq^* D$ means an end-segment of C_δ is contained in D. We write

$$\begin{aligned} (TCG)^* &= \{X \subseteq \omega_1 \mid \exists\, D \text{ club such that } X^*(D) \subseteq X\} \\ (TCG)^+ &= \{X \subseteq \omega_1 \mid \forall\, D \text{ club, we have } X^*(D) \cap X \neq \emptyset\} \\ TCG &= \{X \subseteq \omega_1 \mid \exists\, D \text{ club such that } X^*(D) \cap X = \emptyset\} \end{aligned}$$

We say $\langle C_\delta \mid \delta \in A \rangle$ is *tail club guessing*, if for any club $D \subseteq \omega_1$, $X^*(D) \neq \emptyset$. Notice that $\omega_1 \setminus A$ may or may not be stationary.

Hence we have

$$(TCG)^+ = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in X \cap A \rangle \text{ is tail club guessing}\}$$

The notation $X^*(D)$, TCG and so forth are somewhat abusive. But it will be clear from the context which ladder system and A are under consideration.

3.2 Proposition. $(TCG)^*$ *is a normal filter on ω_1. More technically, we have*

(1) For two clubs D_1 and D_2, $X^*(D_1 \cap D_2) = X^*(D_1) \cap X^*(D_2)$.
(2) For any sequence $\langle D_n \mid n < \omega \rangle$ of clubs, $X^*(\bigcap\{D_n \mid n < \omega\}) \subseteq \bigcap\{X^*(D_n) \mid n < \omega\}$.
(3) For any sequence $\langle D_\xi \mid \xi < \omega_1 \rangle$ of clubs, we have

$$X^*(\{\alpha < \omega_1 \mid \forall\, \xi < \alpha\ \alpha \in D_\xi\}) \subseteq \{\alpha < \omega_1 \mid \forall\, \xi < \alpha\ \alpha \in X^*(D_\xi)\}.$$

(4) $\langle C_\delta \mid \delta \in A \rangle$ is tail club guessing iff $A \in (TCG)^+$ iff $(TCG)^* \subseteq (TCG)^+$ iff $\emptyset \notin (TCG)^*$.

§4. A weak reflection principle introduced

We introduce one of weak reflection principles which are compatible with tail club guessing. What is intended by this principle becomes clear if it gets compared with the second characterization of SRP in §1. We consider its applications and consistency.

4.1 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. We say *(proper, $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, plain)-Reflection Principle* holds, if for any (K, S, θ, a) such that

- $K \supseteq \omega_1$,
- $S \subseteq [K]^\omega$,
- θ is a regular cardinal such that $H_{(2^{|\mathrm{TC}(K)|})^+} \in H_\theta$,
- $a \in H_\theta$,

there exists an $\in$-chain $\langle N_\alpha \mid \alpha < \omega_1\rangle$ in H_θ with $a \in N_0$ and a club $D \subseteq \omega_1$ such that for each $\delta \in X^*(D)$, we have either (1) or (2),

(1) $N_\delta \cap K \in S$.

(2) For any $\in$-chain $\langle N'_n \mid n \leq \omega\rangle$ in H_θ with $\langle N_{\delta_n} \mid n < \omega\rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega\rangle$, we have $N'_\omega \cap K \notin S$, where $\langle \delta_n \mid n < \omega\rangle$ increasingly enumerates C_δ.

§5. A tail club guessing ideal can be saturated, an application

The fact that a tail club guessing ideal can be saturated is due to [I]. Here we show that the same can be said as an application of our weak reflection principle. This use of the reflection principle follows that of SRP in [B] where the nonstationary ideal NS_{ω_1} is shown to be saturated.

5.1 Proposition. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. Assume (proper, $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, plain)-RP. The ideal TCG associated with $\langle C_\delta \mid \delta \in A\rangle$ is either saturated or equals $\mathcal{P}(\omega_1)$ depending on $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing or not, respectively.*

Proof. If $\langle C_\delta \mid \delta \in A\rangle$ is not tail club guessing, then $TCG = \mathcal{P}(\omega_1)$. However, we make no use of $\langle C_\delta \mid \delta \in A\rangle$ being tail club guessing in the rest. Let $\mathcal{B}$ be a maximal antichain in $(TCG)^+$. Let

$$S = \{\overline{N} \prec H_{\omega_2} \mid \exists B \in \overline{N} \cap \mathcal{B} \text{ such that } \overline{N} \cap \omega_1 \in B\}.$$

Apply the assumed reflection principle to $(H_{\omega_2}, S, \theta, \mathcal{B})$. We have an $\in$-chain $\langle N_\alpha \mid \alpha < \omega_1\rangle$ in H_θ and a club D such that for each $\delta \in X^*(D)$, we have (1) or (2),

(1) $N_\delta \cap H_{\omega_2} \in S$. In this case may assume there exists $B \in N_\delta \cap \mathcal{B}$ such that $N_\delta \cap \omega_1 = \delta \in B$.

(2) For any $\in$-chain $\langle N'_n \mid n \leq \omega\rangle$ in H_θ with $\langle N_{\delta_n} \mid n < \omega\rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega\rangle$, we have $N'_\omega \cap H_{\omega_2} \notin S$, where $\langle \delta_n \mid n < \omega\rangle$ increasingly enumerates C_δ.

Claim 1. *$\{\delta \in X^*(D) \mid \delta$ satisfies (1) $\} \in (TCG)^*$ and so it contains $X^*(D_1)$ for some club D_1.*

Proof. By contradiction. Suppose not and let

$$E = \{\delta \in X^*(D) \mid \delta \text{ satisfies (2)}\}.$$

Then $E \in (TCG)^+$. Hence there exists $B \in \mathcal{B}$ with $E \cap B \in (TCG)^+$. This means that

$$\langle C_\delta \mid \delta \in E \cap B\rangle$$

is tail club guessing.

Let $\langle M_\alpha \mid \alpha < \omega_1\rangle$ be an $\in$-chain in H_λ, where λ is sufficiently large, such that $\langle N_\alpha \mid \alpha < \omega_1\rangle, H_\theta, B \in M_0$. Since $\{M_\alpha \cap \omega_1 \mid \alpha < \omega_1\}$ is a club, we have $\delta \in E \cap B$ such that an end-segment of C_δ is contained in this club. By reindexing, we may assume $\{M_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of C_δ. In particular, we have

$$\delta = \sup\{M_n \cap \omega_1 \mid n < \omega\}.$$

Since $\delta \in E$, we know δ satisfies (2). However, since $\langle N_\alpha \mid \alpha < \omega_1 \rangle \in M_n$, we have $N_{M_n \cap \omega_1} \subseteq_{\omega_1} M_n \cap H_\theta$. So $\langle N_{M_n \cap \omega_1} \mid n < \omega \rangle \subseteq_{\omega_1} \langle M_n \cap H_\theta \mid n < \omega \rangle$. Hence we may conclude $M_\omega \cap H_{\omega_2} \notin S$.

Since $B \in (M_\omega \cap H_{\omega_2}) \cap \mathcal{B}$ and $M_\omega \cap \omega_1 = \delta \in B$, we have $M_\omega \cap H_{\omega_2} \in S$. This is a contradiction.

□

Claim 2. *$\mathcal{B} \subseteq \bigcup\{N_\alpha \mid \alpha < \omega_1\}$ and so $|\mathcal{B}| \leq \omega_1$ holds.*

Proof. By claim 1, we have

$$X^*(D_1) \subseteq \{\alpha < \omega_1 \mid \exists B \in N_\alpha \cap \mathcal{B} \text{ such that } \alpha \in B\}.$$

Hence by the normality of TCG, we have $\mathcal{B} \cap \bigcup\{N_\alpha \mid \alpha < \omega_1\}$ is maximal. Hence

$$\mathcal{B} = \mathcal{B} \cap \bigcup\{N_\alpha \mid \alpha < \omega_1\} \subseteq \bigcup\{N_\alpha \mid \alpha < \omega_1\}.$$

□

□

§6. Getting our weak reflection principle from a forcing axiom

Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. We consider a class of preorders with the ladder system as a parameter. If the ladder system is tail club guessing then this tail club guessing remains in any generic extension by any preorder in this class. This class of preorders are iterable under our revised countable support iterations when combined with semiproper. Notice that this class of preorders may contain preorders which are not semiproper.

6.1 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. We say a preorder P is $\langle C_\delta \mid \delta \in A\rangle$-*$\omega$-semiproper*, if for all sufficiently large regular cardinals λ and all $\in$-chains $\langle M_n \mid n < \omega\rangle$ in H_λ such that $\langle C_\delta \mid \delta \in A\rangle, P \in M_0$, $M_\omega \cap \omega_1 \in A$ and $\{M_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of $C_{M_\omega \cap \omega_1}$, if $p \in P \cap M_0$, then there exists $q \leq p$ in P such that for all $n < \omega$, q is (P, M_n)-semi-generic.

6.2 Lemma. *Let (K, S, θ, a) be as in our reflection principle in §4. Then there exists a p.o. set P such that*

(1) P is proper and σ-Baire,

(2) P is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper,

(3) P adds a sequence $\langle \dot{N}_\alpha \mid \alpha < \omega_1\rangle$ such that every initial segment of this is an $\in$-chain in H_θ^V and also adds a club $\dot{D}$ such that for each $\delta \in X^*(\dot{D})$, we have either (3.1) or (3.2),

(3.1) $\dot{N}_\delta \cap K \in S$.

(3.2) For any $\in$-chain $\langle N'_n \mid n \leq \omega\rangle \in V$ in H_θ^V with $\langle \dot{N}_{\delta_n} \mid n < \omega\rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega\rangle$, we have $N'_\omega \cap K \notin S$, where $\langle \delta_n \mid n < \omega\rangle$ increasingly enumerates C_δ.

Proof. Let $p = \big(\langle N^p_\alpha \mid \alpha \leq \alpha^p\rangle,\ D^p\big) \in P$, if

- $\alpha^p < \omega_1$,
- $\langle N^p_\alpha \mid \alpha \leq \alpha^p\rangle$ is an $\in$-chain in H_θ with $a \in N^p_0$,
- $D^p \subseteq \alpha^p + 1$ is closed,
- For any $\delta \leq \alpha^p$ with $\delta \in A$, if $C_\delta \subseteq^* D^p$, then we have (1) or (2),

  (1) $N^p_\delta \cap K \in S$.

(2) For any $\in$-chain $\langle N'_n \mid n \leq \omega \rangle$ in H_θ with $\langle N^p_{\delta_n} \mid n < \omega \rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega \rangle$, we have $N'_\omega \cap K \notin S$, where $\langle \delta_n \mid n < \omega \rangle$ increasingly enumerates C_δ.

For $p, q \in P$, let $q \leq p$ in P, if

$$\langle N^q_\alpha \mid \alpha \leq \alpha^q \rangle \supseteq \langle N^p_\alpha \mid \alpha \leq \alpha^p \rangle,$$

$$D^q \cap (\alpha^p + 1) = D^p.$$

Claim 1. (Dense) *For any $\xi < \omega_1$, $p \in P$ with $\alpha^p < \xi$, $e \in H_\theta$, there exists $q \in P$ such that $\alpha^q = \xi$, $e \in N^q_\xi$, and $D^q = D^p \cup \{\xi\}$.*

Proof. Construct an $\in$-chain $\langle N^q_\alpha \mid \alpha \leq \xi \rangle$ which just extends $\langle N^p_\alpha \mid \alpha \leq \alpha^p \rangle$ and $e \in N^q_\xi$. Then set $D^q = D^p \cup \{\xi\}$. Then this q works. Notice that if $\delta \leq \xi, \delta \in A$ and $C_\delta \subseteq^* D^q$, then $\delta \leq \alpha^p$ and $C_\delta \subseteq^* D^p$. So there is no new δ to worry about.

□

For (1): We show P is proper by ESCAPE. Let λ be a sufficiently large regular cardinal and $M \prec H_\lambda$ with $P \in M$. Let $p \in P \cap M$.

Claim 2. *There exists $q \leq p$ such that q in (P, M)-generic, $\alpha^q = M \cap \omega_1 \in D^q$ and $N^q_{\alpha^q} = M \cap H_\theta$.*

Proof. Let $\langle D_n \mid n < \omega \rangle$ enumerate the dense subsets $D \in M$ of P. Construct p_n and p'_n by recursion such that for each $n < \omega$, we have

- $p_0 = p$,
- $p_n \in P \cap M$, $p_n \leq p$,
- $p'_n < p_n$, $p'_n \in P \cap M$,
- $D^{p'_n} = D^{p_n} \cup \{\alpha^{p'_n}\}$,
- $(\alpha^{p_n}, \alpha^{p'_n}) \cap C_{M \cap \omega_1} \neq \emptyset$, if $M \cap \omega_1 \in A$,
- $p_{n+1} \leq p'_n$ and $p_{n+1} \in D_n \cap M$.

Now let

$$\langle N^q_\alpha \mid \alpha \leq M \cap \omega_1 \rangle = (\bigcup \{\langle N^{p_n}_\alpha \mid \alpha \leq \alpha^{p_n} \rangle \mid n < \omega\}) \cup \{(M \cap \omega_1, M \cap H_\theta)\},$$

$$D^q = \bigcup \{D^{p_n} \mid n < \omega\} \cup \{M \cap \omega_1\}.$$

Then $q \in P$ and is (P, M)-generic. Notice that if $\delta \leq M \cap \omega_1, \delta \in A$ and $C_\delta \subseteq^* D^q$, then $\delta < M \cap \omega_1$.

By the above argument, we see that P is σ-Baire, too. Namely, P adds no new ω-sequences of ordinals.

For (2): We show P is $\langle C_\delta \mid \delta \in A \rangle$-$\omega$-semiproper. Let λ be a sufficiently large regular cardinal and $\langle M_n \mid n \leq \omega \rangle$ be an $\in$-chain in H_λ with $\langle C_\delta \mid \delta \in A \rangle, P \in M_0$, $M_\omega \cap \omega_1 \in A$ and $\{M_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of $C_{M_\omega \cap \omega_1}$. Let $p \in P \cap M_0$.

Claim 3. *There exists $q \leq p$ such that for all $n < \omega$, q is (P, M_n)-semi-generic.*

Proof. We argue in two cases.

Case 1. For any $\in$-chain $\langle M'_n \mid n \leq \omega \rangle$ in H_λ with $\langle M_n \mid n < \omega \rangle \subseteq_{\omega_1} \langle M'_n \mid n < \omega \rangle$, we have $M'_\omega \cap K \notin S$: Since $H_{|\mathrm{TC}(K)|^+} \in H_\theta \in H_\lambda$, by 2.2 corollary (sequential three H lemma), we have:

For any $\in$-chain $\langle \overline{M}_n \mid n \leq \omega \rangle$ in H_θ with $\langle M_n \cap H_\theta \mid n < \omega \rangle \subseteq_{\omega_1} \langle \overline{M}_n \mid n < \omega \rangle$, we have $\overline{M}_\omega \cap K \notin S$.

We construct $\langle q_n \mid n < \omega \rangle$ by recursion such that for each $n < \omega$, we have

- $q_0 < p$, q_0 is (P, M_0)-generic, $\alpha^{q_0} = M_0 \cap \omega_1 \in D^{q_0}$, $N^{q_0}_{\alpha^{q_0}} = M_0 \cap H_\theta$ and $q_0 \in M_1$.
- q_n is (P, M_n)-generic, $\alpha^{q_n} = M_n \cap \omega_1 \in D^{q_n}$, $N^{q_n}_{\alpha^{q_n}} = M_n \cap H_\theta$ and $q_n \in M_{n+1}$.
- $q_{n+1} < q_n$.

Let
$$\langle N^q_\alpha \mid \alpha \leq M_\omega \cap \omega_1 \rangle = (\bigcup \{\langle N^{q_n}_\alpha \mid \alpha \leq \alpha^{q_n} \rangle \mid n < \omega\}) \cup \{(M_\omega \cap \omega_1, M_\omega \cap H_\theta)\},$$
$$D^q = (\bigcup \{D^{q_n} \mid n < \omega\}) \cup \{M_\omega \cap \omega_1\}.$$

Then $q \in P$ and is (P, M_n)-generic and so (P, M_n)-semi-generic for all $n < \omega$. Notice that
$$C_{M_\omega \cap \omega_1} \subseteq^* \{M_n \cap \omega_1 \mid n < \omega\} \subseteq D^q$$
and we have,

For any $\in$-chain $\langle N'_n \mid n \leq \omega \rangle$ in H_θ with $\langle N^q_{\delta_n} \mid n < \omega \rangle \subseteq_{\omega_1} \langle N'_n \mid n < \omega \rangle$, we have $N'_\omega \cap K \notin S$.

Case 2. There exists an $\in$-chain $\langle M'_n \mid n < \omega \rangle$ in H_λ such that $\langle M_n \mid n < \omega \rangle \subseteq_{\omega_1} \langle M'_n \mid n < \omega \rangle$ and $M'_\omega \cap K \in S$:

We construct $\langle q_n \mid n < \omega \rangle$ by recursion such that for each $n < \omega$, we have

- $q_0 < p$, q_0 is (P, M'_0)-generic and so is (P, M_0)-semi-generic, $\alpha^{q_0} = M'_0 \cap \omega_1 = M_0 \cap \omega_1 \in D^{q_0}$, $N^{q_0}_{\alpha^{q_0}} = M'_0 \cap H_\theta$ and $q_0 \in M'_1$,
- q_n is (P, M'_n)-generic and so is (P, M_n)-semi-generic, $\alpha^{q_n} = M'_n \cap \omega_1 = M_n \cap \omega_1 \in D^{q_n}$, $N^{q_n}_{\alpha^{q_n}} = M'_n \cap H_\theta$ and $q_n \in M'_{n+1}$,
- $q_{n+1} < q_n$.

Let
$$\langle N^q_\alpha \mid \alpha \leq M_\omega \cap \omega_1 \rangle = (\bigcup \{\langle N^{q_n}_\alpha \mid \alpha \leq \alpha^{q_n} \rangle \mid n < \omega\}) \cup \{(M'_\omega \cap \omega_1, M'_\omega \cap H_\theta)\},$$
$$D^q = (\bigcup \{D^{q_n} \mid n < \omega\}) \cup \{M'_\omega \cap \omega_1\}.$$

Then $q \in P$ and is (P, M'_n)-generic and so is (P, M_n)-semi-generic for all $n < \omega$. Notice that
$$C_{M'_\omega \cap \omega_1} = C_{M_\omega \cap \omega_1} \subseteq^* \{M_n \cap \omega_1 \mid n < \omega\} \subseteq D^q$$
and we have
$$N^q_{M'_\omega \cap \omega_1} \cap K = N^q_{M_\omega \cap \omega_1} \cap K = M'_\omega \cap K \in S.$$

For (3): Let G be P-generic over V. Let
$$\langle \dot N_\alpha \mid \alpha < \omega_1 \rangle = \bigcup \{\langle N^p_\alpha \mid \alpha \leq \alpha^p \rangle \mid p \in G\},$$
$$\dot D = \bigcup \{D^p \mid p \in G\}.$$

Then by genericity these two work and we are done.

□

§7. The nonstationary ideal may not be saturated under tail club guessing

The following is implicit in [I].

7.1 Theorem. *Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system on A which may or may not be tail club guessing but assume A is stationary.*

(1) If MA^+(Adding a club subset of ω_1 by countable conditions) holds, then

$$\{S \subseteq A \mid S \text{ is stationary and } \langle C_\delta \mid \delta \in S\rangle \text{ fails to be tail club guessing i.e., } S \in TCG\}$$

is dense below A in $\mathcal{P}(\omega_1)/NS_{\omega_1}$. Hence

$A \Vdash_{\mathcal{P}(\omega_1)/NS_{\omega_1}}$ "$\exists S \in \dot{G}$, $S \subseteq A$ *and* $((\langle C_\delta \mid \delta \in S\rangle$ *fails to be club guessing*$)^V$".

(2) If MA^+(Adding a club subset of ω_1 by countable conditions) holds, then there exist $\langle S_i \mid i < \mu\rangle$ and $\langle D_i \mid i < \mu\rangle$ such that

- *$\langle S_i \mid i < \mu\rangle$ lists stationary subsets of A one-to-one manner and $\{S_i \mid i < \mu\}$ is a maximal antichain below A in $\mathcal{P}(\omega_1)/NS_{\omega_1}$,*
- *D_i is a club subset of ω_1,*
- *$S_i \subseteq \{\delta \in A \mid C_\delta \not\subseteq^* D_i\}$.*

(3) If in addition we assume $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing, then $|\mu| \geq \omega_2$.

Proof. Let $E \subseteq A$ be stationary. Let P be the notion of forcing which adds a club $\dot{D}$ by initial segments. Then P is σ-closed. By genericity,

Claim 1. $\Vdash_P$ "$\dot{S} = \{\delta \in E \mid C_\delta \not\subseteq^* \dot{D}\}$ *is stationary".*

Proof. Supose $p \Vdash_P$"$\dot{C} \subseteq \omega_1$ is a club". Let λ be a sufficiently large regular cardinal and take $M \prec H_\lambda$ such that $p, \dot{C}, P \in M$ and $M \cap \omega_1 = \delta \in E$. Easy to construct $q < p$ such that

- q is (P, M)-generic,
- $q \Vdash_P$"$C_\delta \not\subseteq^* \dot{D}$",

Since $q \Vdash_P$"$\delta = M \cap \omega_1 = M[\dot{D}] \cap \omega_1 \in \dot{C}$", we are done.

□

For (1): Apply MA$^+$(Adding a club subset of ω_1 by countable conditions). Get a filter $F \subset P$ which is generic over relevant ω_1-many dense subsets of P. Let

$$D_F = \{\delta < \omega_1 \mid \exists p \in F\ p \Vdash_P \text{“}\delta \in \dot{D}\text{”}\} = \bigcup F,$$

$$S_F = \{\delta < \omega_1 \mid \exists p \in F\ p \Vdash_P \text{“}\delta \in \dot{S}\text{”}\}.$$

Then we may assume D_F is a club. By MA$^+$, S_F is stationary. We want and may arrange

$$S_F = \{\delta \in E \mid C_\delta \not\subseteq^* D_F\}.$$

Here are some details. For $\delta \in E$, let

$$D(\delta) = \{p \in P \mid p \Vdash_P \text{“}C_\delta \subseteq^* \dot{D}\text{” or } p \Vdash_P \text{“}C_\delta \not\subseteq^* \dot{D}\text{”}\}.$$

Make sure F hits these $D(\delta)$'s.

We may also prepare ω_1-many functions $\langle n \mapsto \dot{m}(\delta, n) \mid n < \omega\rangle$ in V^P such that

- If $C_\delta \not\subseteq^* \dot{D}$, then for all $n < \omega$, we have $n < \dot{m}(\delta, n) < \omega$ and $C_\delta(\dot{m}(\delta, n)) \notin \dot{D}$.
- If $C_\delta \subseteq^* \dot{D}$, then $\dot{m}(\delta, n) = \dot{m}(\delta)$ constantly and we have $C_\delta\lceil[\dot{m}(\delta), \omega) \subset \dot{D}$.

where $\langle C_\delta(n) \mid n < \omega\rangle$ increasingly enumerates C_δ and $C_\delta\lceil[m, \omega) = \{C_\delta(k) \mid m \leq k < \omega\}$.

Let F decide the functions to $\langle n \mapsto m(\delta, n) \mid n < \omega\rangle$.

Now suppose $\delta \in S_F$. Then $p \Vdash_P$"$C_\delta \not\subseteq^* \dot{D}$" for some $p \in F$. Hence there exists $p' \in F$ such that for all $n < \omega$, we have $p' \Vdash_P$"$n < \dot{m}(\delta, n) < \omega$, $C_\delta(\dot{m}(\delta, n)) \notin \dot{D}$". Therefore, we may conclude $C_\delta(m(\delta, n)) \notin D_F$ and $n < m(\delta, n)$. Hence $C_\delta \not\subseteq^* D_F$.

Conversely, suppose $C_\delta \not\subseteq^* D_F$. Then $p \Vdash_P$"$C_\delta \not\subseteq^* \dot{D}$" for some $p \in F$. This is because if $p \Vdash_P$"$C_\delta \subseteq^* \dot{D}$", then $p \Vdash_P$"$C_\delta \lceil [\dot{m}(\delta), \omega) \subseteq \dot{D}$" and so $C_\delta \lceil [m(\delta), \omega) \subset D_F$. This is a contradiction.

$p \Vdash_P$"$C_\delta \not\subseteq^* \dot{D}$" in turn implies $p \Vdash_P$"$\delta \in \dot{S}$" and so $\delta \in S_F$.

For (2): By (1), every statioanry subset below A gets extended to some stationary S with $S \in TCG$. Hence we may construct $\langle S_i \mid i < \mu \rangle$ as specified for some μ. Since $S_i \in TCG$, $\langle C_\delta \mid \delta \in S_i \rangle$ fails to be tail club guessing. Take a club D_i such that $S_i \subseteq \{\delta \in E \mid C_\delta \not\subseteq^* D_i\}$.

For (3): By contradiction. Since case $\mu \leq \omega$ is similar, we assume $\mu = \omega_1$. Let D be the diagonal intersection of the D_i. Since $\langle C_\delta \mid \delta \in A \rangle$ is tail club guessing, we have $\{\delta \in A \mid C_\delta \subseteq^* D\}$ is stationary. Since the S_i's are maximal below A, we have some S_{i_0} such that $\{\delta \in S_{i_0} \mid C_\delta \subseteq^* D\}$ is stationary. Pick $\delta \in S_{i_0}$ with $i_0 < \delta$ and $C_\delta \subseteq^* D$. Then for all $n < \omega$ with $i_0 < C_\delta(n) \in D$, we have $C_\delta(n) \in D_{i_0}$. Hence $C_\delta \subseteq^* D_{i_0}$. This contradicts to the choice of $\delta \in S_{i_0}$.

□

7.2 Corollary. *(1) (MA$^+$(σ-closed p.o. sets)) If NS_{ω_1} is saturated, then no tail club guessing holds.*
(2) (MA$^+$(σ-closed p.o. sets)) If SRP holds, then no tail club guessing holds.

7.3 Note. If we have ω-semiproper p.o. sets which iteratively force SRP, then MA$^+$(ω-semiproper) would imply that SRP and any tail club guessing in the ground model would remain. Hence we would get MA$^+$(σ-closed p.o. sets), SRP and tail club guessing. This is a contradiction. Hence it is quite unlikely to have ω-semiproper p.o. sets which iteratively force SRP.

7.4 Question. Does SRP negate tail club guessing ?

§8. Trees of clubs and ω-stationary sets

We know by [S] proper and semiproper preorders are characterized in terms of preserving the stationary sets and the semistationary sets, respectively. We would like to consider a characterization of $\langle C_\delta \mid \delta \in A \rangle$-$\omega$-semiproper preorders along this line. We introduce trees of clubs in $[K]^\omega$ and ω-stationary sets in $Seq^\omega(K)$. They are counterparts to clubs in $[K]^\omega$ and stationary sets in $[K]^\omega$.

8.1 Notation. Let K be a set with $K \supseteq \omega_1$. Let $h : [K]^{<\omega} \longrightarrow K$ be a map. Then for $a \in [K]^\omega$, we say *a is h-closed*, if for all $x \in [a]^{<\omega}$, we have $h(x) \in a$. We denote

$$C(h) = \{a \in [K]^\omega \mid a \text{ is } h\text{-closed}\}.$$

Then $C(h)$ is a club in $[K]^\omega$. It is well-known that every club in $[K]^\omega$ contains $C(h)$ for some h.

8.2 Definition. Let K be a set with $K \supseteq \omega_1$. Denote

$$Seq^\omega(K) = \{\langle a_n \mid n < \omega \rangle \mid \text{for all } n < \omega,\ a_n \in [K]^\omega\}.$$

We call f is a *tree of clubs in* $[K]^\omega$, if f is a function such that

- $Dom(f) = {}^{<\omega}([K]^\omega)$,
- For all $\langle a_0, \cdots, a_{n-1} \rangle \in Dom(f)$, we have $f_{\langle a_0, \cdots, a_{n-1} \rangle}$ is a function from $[K]^{<\omega}$ to K.

For a tree of clubs f in $[K]^\omega$, denote

$$B(f) = \{\langle a_n \mid n < \omega \rangle \in Seq^\omega(K) \mid \text{ for all } n < \omega,\ a_n \text{ is } f_{\langle a_0, \cdots, a_{n-1} \rangle}\text{-closed}\}.$$

Let $S \subseteq Seq^\omega(K)$. We say S is *ω-stationary in $Seq^\omega(K)$*, if for all trees of clubs f in $[K]^\omega$, we have

$$B(f) \cap S \neq \emptyset.$$

We say S is *ω-semistationary in $Seq^\omega(K)$*, if

$$S^* = \{\langle b_n \mid n < \omega\rangle \in Seq^\omega(K) \mid \text{ there exists } \langle a_n \mid n<\omega\rangle \in S \text{ such that for all } n<\omega,\ a_n \subseteq_{\omega_1} b_n\}$$

is ω-stationary in $Seq^\omega(K)$, where $a_n \subseteq_{\omega_1} b_n$ means $a_n \subseteq b_n$ and $a_n \cap \omega_1 = b_n \cap \omega_1$.

If we write

$$\langle a_n \mid n<\omega\rangle \subseteq_{\omega_1} \langle b_n \mid n<\omega\rangle,$$

then this means that for all $n<\omega$, we have $a_n \subseteq b_n$ and $a_n \cap \omega_1 = b_n \cap \omega_1$.

8.3 Definition. Let $\omega_1 \subseteq K_1 \subseteq K_2$. For $S \subseteq Seq^\omega(K_1)$, define

$$S \uparrow K_2 = \{\langle b_n \mid n<\omega\rangle \in Seq^\omega(K_2) \mid \langle b_n \cap K_1 \mid n<\omega\rangle \in S\}.$$

For $T \subseteq Seq(K_2)$, define

$$T \downarrow K_1 = \{\langle b_n \cap K_1 \mid n<\omega\rangle \in Seq^\omega(K_1) \mid \langle b_n \mid n<\omega\rangle \in T\}.$$

We may call $S\uparrow K_2$ the *lift-up* of S to K_2 and $T \downarrow K_1$ the *pull-down* of T to K_1. We have

$$(S\uparrow K_2)\downarrow K_1 = S,\ (T\downarrow K_1)\uparrow K_2 \supseteq T.$$

So we loose some specificness if we first go down and then go up.

8.4 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. Denote

$$\mathcal{E} = \{C_\delta\lceil[m,\omega) \in Seq^\omega(\omega_1) \mid \delta \in A,\ m<\omega\},$$

where $C_\delta\lceil[m,\omega) = \{(n-m, C_\delta(n)) \mid m \le n < \omega\}$ and $C_\delta(n)$ denotes the n-th element of C_δ.

For any set $K \supseteq \omega_1$, the lift-up of $\mathcal{E}$ to K is

$$\mathcal{E}\uparrow K = \{\langle a_n \mid n<\omega\rangle \in Seq^\omega(K) \mid \langle a_n \cap \omega_1 \mid n<\omega\rangle \in \mathcal{E}\}.$$

The behavior of $\mathcal{E}$ is the heart of all in this paper. Notice that the ladder system $\langle C_\delta \mid \delta\in A\rangle$ gets recovered from $\mathcal{E}$ in an obvious manner, if $C_\delta(n) \ge \omega$ for all $\delta \in A$ and all $n<\omega$.

§9. Trees of clubs going up and down and their diagonals

Basics with trees of clubs.

9.1 Lemma. (Typical trees of clubs) *Let θ be a regular uncountable cardinal so that $[H_\theta]^\omega \subset H_\theta$ and $x \in H_\theta$. Then there exists a tree of clubs f in $[H_\theta]^\omega$ such that*

$$B(f) \subseteq \{\langle N_n \mid n<\omega\rangle \in Seq^\omega(H_\theta) \mid \langle N_n \mid n<\omega\rangle \text{ is an } \in\text{-chain in } H_\theta \text{ with } x \in N_0\}.$$

Proof. For $\langle N_0,\cdots,N_{n-1}\rangle \in {}^{<\omega}([H_\theta]^\omega)$, let

$$f_{\langle N_0,\cdots,N_{n-1}\rangle} : [H_\theta]^{<\omega} \longrightarrow H_\theta$$

be such that

$$C(f_{\langle N_0,\cdots,N_{n-1}\rangle}) \subseteq \{N \prec H_\theta \mid x, \langle N_0,\cdots,N_{n-1}\rangle \in N\}.$$

Then this tree of clubs f works.

□

9.2 Lemma. (Going up) *Let $\omega_1 \subseteq K_1 \subseteq K_2$. If g is a tree of clubs in $[K_1]^\omega$, then there exists a tree of clubs f in $[K_2]^\omega$ such that*

$$B(f) \subseteq B(g) \uparrow K_2.$$

Proof. For $\langle b_0, \cdots, b_{n-1}\rangle \in {}^{<\omega}([K_2]^\omega)$ with $\langle b_0 \cap K_1, \cdots, b_{n-1} \cap K_1\rangle \in {}^{<\omega}([K_1]^\omega)$, let

$$f_{\langle b_0,\cdots,b_{n-1}\rangle} : [K_2]^{<\omega} \longrightarrow K_2$$

be such that

$$C(f_{\langle b_0,\cdots,b_{n-1}\rangle}) \subseteq \{b \in [K_2]^\omega \mid \exists a \in C(g_{\langle b_0\cap K_1,\cdots,b_{n-1}\cap K_1\rangle})\ b \cap K_1 = a\}.$$

Claim. $B(f) \subseteq B(g) \uparrow K_2 = \{\langle b_n \mid n < \omega\rangle \in Seq^\omega(K_2) \mid \langle b_n \cap K_1 \mid n < \omega\rangle \in B(g)\}$.

Proof. Let $\langle b_n \mid n < \omega\rangle \in B(f)$. Then

$$b_n \in C(f_{\langle b_0,\cdots,b_{n-1}\rangle}).$$

Hence by induction on n, we have

$$b_n \cap K_1 \in C(g_{\langle b_0\cap K_1,\cdots,b_{n-1}\cap K_1\rangle}).$$

Therefore

$$\langle b_n \cap K_1 \mid n < \omega\rangle \in B(g).$$

□

9.3 Lemma. (Going down) *Let $\omega_1 \subseteq K_1 \subseteq K_2$. If f is a tree of clubs in $[K_2]^\omega$, then there exists a tree of clubs g in $[K_1]^\omega$ such that*

$$B(g) \subseteq B(f) \downarrow K_1.$$

Proof. For $a \in [K_1]^\omega$ and $\emptyset$, $\langle a_0, \cdots, a_n\rangle \in {}^{<\omega}([K_1]^\omega)$, we define $\hat{a}^{(\emptyset)}$, $\hat{a}^{(\langle a_0,\cdots,a_n\rangle)} \in [K_2]^\omega$ by recursion on n as follows;

$$\hat{a}^{(\emptyset)} = \text{ the } \subseteq\text{-least } f_\emptyset\text{-closed set } b \text{ with } b \supseteq a,$$

$$\hat{a}^{(\langle a_0,\cdots,a_n\rangle)} = \text{ the } \subseteq\text{-least } f_{\langle \hat{a}_0^{(\emptyset)},\cdots,\hat{a}_n^{(\langle a_0,\cdots,a_{n-1}\rangle)}\rangle}\text{-closed set } b \text{ with } b \supseteq a.$$

We choose a tree of clubs g in $[K_1]^\omega$ so that

$$C(g_\emptyset) \subseteq \{b \cap K_1 \mid b \in C(f_\emptyset)\},$$

$$C(g_{\langle a_0,\cdots,,a_n\rangle}) \subseteq \{b \cap K_1 \mid b \in C(f_{\langle \hat{a}_0^{(\emptyset)},\cdots,\hat{a}_n^{(\langle a_0,\cdots,a_{n-1}\rangle)}\rangle})\}.$$

Claim. $B(g) \subseteq B(f) \downarrow K_1$.

Proof. Let $\langle a_n \mid n < \omega\rangle \in B(g)$. Then a_0 is $g_\emptyset$-closed. So there exists a_0' which is $f_\emptyset$-closed and

$$a_0 = a_0' \cap K_1.$$

Since $\hat{a}_0^{(\emptyset)}$ is the $f_\emptyset$-closure of a_0, we have

$$a_0 \subseteq \hat{a}_0^{(\emptyset)} \subseteq a_0'.$$

And so

$$a_0 \subseteq \hat{a}_0^{(\emptyset)} \cap K_1 \subseteq a_0' \cap K_1 = a_0.$$

Hence

$$a_0 = \hat{a}_0^{(\emptyset)} \cap K_1.$$

Similarly, a_{n+1} is $g_{\langle a_0,\cdots,a_n\rangle}$-closed. So there exists a_{n+1}' which is $f_{\langle \hat{a}_0^{(\emptyset)},\cdots,\hat{a}_n^{(\langle a_0,\cdots,a_{n-1}\rangle)}\rangle}$-closed and

$$a_{n+1} = a_{n+1}' \cap K_1.$$

Since $\hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)}$ is the $f_{\langle \hat{a}_0^{(\emptyset)},\cdots,\hat{a}_n^{(\langle a_0,\cdots,a_{n-1}\rangle)}\rangle}$-closure of a_{n+1}, we have

$$a_{n+1} \subseteq \hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)} \subseteq a_{n+1}'.$$

And so

$$a_{n+1} \subseteq \hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)} \cap K_1 \subseteq a_{n+1}' \cap K_1 = a_{n+1}.$$

Hence

$$a_{n+1} = \hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)} \cap K_1.$$

By the definition of $\hat{a}^{(\langle a_0,\cdots,a_n\rangle)}$'s, we have

$$\hat{a}_0^{(\emptyset)} \in C(f_\emptyset),$$

$$\hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)} \in C(f_{\langle \hat{a}_0^{(\emptyset)},\cdots,\hat{a}_n^{(\langle a_0,\cdots,a_{n-1}\rangle)}\rangle}).$$

Hence

$$\langle \hat{a}_0^{(\emptyset)}, \hat{a}_1^{(\langle a_0\rangle)}, \hat{a}_2^{(\langle a_0,a_1\rangle)}, \cdots, \hat{a}_{n+1}^{(\langle a_0,\cdots,a_n\rangle)}, \cdots \rangle \in B(f).$$

Therefore, $\langle a_n \mid n < \omega\rangle \in B(f) \downarrow K_1$.

□

9.4 Lemma. (σ-closed) If f^k are trees of clubs in $[K]^\omega$ for all $k < \omega$, then there exists a tree of clubs f in $[K]^\omega$ such that

$$B(f) \subseteq \bigcap\{B(f^k) \mid k < \omega\}.$$

Proof. For $\langle a_0,\cdots,a_{n-1}\rangle \in {}^{<\omega}([K]^\omega)$, define $f_{\langle a_0,\cdots,a_{n-1}\rangle} : [K]^{<\omega} \longrightarrow K$ so that

$$C(f_{\langle a_0,\cdots,a_{n-1}\rangle}) \subseteq \bigcap\{C(f^k_{\langle a_0,\cdots,a_{n-1}\rangle}) \mid k < \omega\}.$$

Then this f works.

□

9.5 Lemma. (Diagonal intersection) *Let f^v be a tree of clubs in $[K]^\omega$ for all $v \in K$. Then there exists a tree of clubs f in $[K]^\omega$ such that for all $\langle a_0,\cdots,a_{n-1}\rangle \in {}^{<\omega}([K]^\omega)$, we have*

$$C(f_{\langle a_0,\cdots,a_{n-1}\rangle}) \subseteq \{a \in [K]^\omega \mid \forall v \in a \; a \in C(f^v_{\langle a_0,\cdots,a_{n-1}\rangle})\}.$$

Proof. Let $\langle a_0,\cdots,a_{n-1}\rangle \in {}^{<\omega}([K]^\omega)$. For all $v \in K$, we form clubs $C(f^v_{\langle a_0,\cdots,a_{n-1}\rangle})$ in $[K]^\omega$. Then take their diagonal intersection. So we have a function

$$f_{\langle a_0,\cdots,a_{n-1}\rangle} : [K]^{<\omega} \longrightarrow K$$

such that

$$C(f_{\langle a_0,\cdots,a_{n-1}\rangle}) \subseteq \{a \in [K]^\omega \mid \forall v \in a\ a \in C(f^v_{\langle a_0,\cdots,a_{n-1}\rangle})\}.$$

□

§10. ω-Stationary sets going up and down and Fodor's lemma

Basics with ω-stationary sets.

10.1 Lemma. (ω-stationary sets going up and down) *Let $\omega_1 \subseteq K_1 \subseteq K_2$. Then we have*

(1) If $S \subseteq Seq^\omega(K_1)$ is ω-stationary, then

$$S \uparrow K_2 = \{\langle b_n \mid n < \omega\rangle \in Seq^\omega(K_2) \mid \langle b_n \cap K_1 \mid n < \omega\rangle \in S\}$$

is ω-stationary in $Seq^\omega(K_2)$.

(2) If $T \subseteq Seq^\omega(K_2)$ is ω-stationary, then

$$T \downarrow K_1 = \{\langle b_n \cap K_1 \mid n < \omega\rangle \in Seq^\omega(K_1) \mid \langle b_n \mid n < \omega\rangle \in T\}$$

is ω-stationary in $Seq^\omega(K_1)$.

Proof. By 9.2 lemma (going up) and 9.3 lemma (going down).

□

10.2 Lemma. (Fodor's lemma) *Let $S \subseteq Seq^\omega(K)$ be ω-stationary in $Seq^\omega(K)$ and r be a map from S to K such that for all $\langle a_n \mid n < \omega\rangle \in S$, we have*

$$r(\langle a_n \mid n < \omega\rangle) \in a_0.$$

Then there exists $S' \subseteq S$ and $v \in K$ such that S' is ω-stationary in $Seq^\omega(K)$ and for all $\langle a_n \mid n < \omega\rangle \in S'$, we have

$$r(\langle a_n \mid n < \omega\rangle) = v.$$

Proof. By contradiction. Suppose for all $v \in K$, the preimages $r^{-1}(\{v\})$ of $\{v\}$ are not ω-stationary in $Seq^\omega(K)$. Take a tree of clubs f^v in $[K]^\omega$such that

$$B(f^v) \cap \{\langle a_n \mid n < \omega\rangle \in S \mid r(\langle a_n \mid n < \omega\rangle) = v\} = \emptyset.$$

Let f be the diagonal intersection of the f^v's so that for all $\langle a_0, \cdots, a_{n-1}\rangle \in {}^{<\omega}([K]^\omega)$,

$$C(f_{\langle a_0,\cdots,a_{n-1}\rangle}) \subseteq \{a \in [K]^\omega \mid \forall v \in a\ a \in C(f^v_{\langle a_0,\cdots,a_{n-1}\rangle})\}.$$

Since for any $a \in [K]^\omega$,

$$\{b \in [K]^\omega \mid a \subseteq b\}$$

is a club in $[K]^\omega$, we may assume, if $\langle a_n \mid n < \omega\rangle \in B(f)$, then

$$a_0 \subseteq a_1 \subseteq a_2 \subseteq \cdots \subseteq a_n \subseteq \cdots.$$

Since S is ω-stationary, $B(f) \cap S \neq \emptyset$. Take $\langle a_n \mid n < \omega\rangle$ such that

$$\langle a_n \mid n < \omega\rangle \in B(f) \cap S.$$

Let $v_0 \in K$ be such that
$$r(\langle a_n \mid n < \omega \rangle) = v_0.$$
Then
$$v_0 \in a_0.$$
Let $n < \omega$. Since $a_0 \subseteq a_n$, we have
$$v_0 \in a_n.$$
Since $a_n \in C(f_{\langle a_0, \cdots, a_{n-1} \rangle})$, we have
$$a_n \in C(f^{v_0}_{\langle a_0, \cdots, a_{n-1} \rangle}).$$
Hence
$$\langle a_n \mid n < \omega \rangle \in B(f^{v_0}).$$
Since $\langle a_n \mid n < \omega \rangle \in B(f^{v_0}) \cap r^{-1}(\{v_0\}) = \emptyset$, this is a contradiction.

□

§11. Trees of clubs vs. clubs and ω-stationary vs. stationary

We may say that $Seq^{\omega}(K)$ is more complex than $[K]^{\omega}$. It appears trees of clubs in $[K]^{\omega}$ are more complex than clubs in $[K]^{\omega}$. Also ω-stationary sets in $Seq^{\omega}(K)$ are more complex than stationary sets in $[K]^{\omega}$. This leads us to consider a map which associate $\bigcup\{a_n \mid n < \omega\} \in [K]^{\omega}$ for each $\langle a_n \mid n < \omega \rangle \in Seq^{\omega}(K)$. We may call this map, the projection $P : Seq^{\omega}(K) \longrightarrow [K]^{\omega}$.

11.1 Lemma. *($Seq^{\omega}(K) \longrightarrow [K]^{\omega}$) Let $K \supseteq \omega_1$. Then*

(1) For any club $C \subseteq [K]^{\omega}$, there exists a tree of clubs f in $[K]^{\omega}$ such that its projection to $[K]^{\omega}$
$$P(f) = P(B(f)) = \{a_{\omega} \in [K]^{\omega} \mid \langle a_n \mid n < \omega \rangle \in B(f), a_{\omega} = \bigcup\{a_n \mid n < \omega\}\}$$
is a subset of C.

(2) If S is ω-stationary in $Seq^{\omega}(K)$, then its projection
$$P(S) = \{a_{\omega} \in [K]^{\omega} \mid \langle a_n \mid n < \omega \rangle \in S, a_{\omega} = \bigcup\{a_n \mid n < \omega\}\}$$
to $[K]^{\omega}$ is stationary in $[K]^{\omega}$.

Proof. Easy.

□

§12. Relativizations and tail club guessing

We consider a connection between ω-stationary sets in $Seq^{\omega}(\omega_1)$ and ladder systems $\langle C_\delta \mid \delta \in A \rangle$ which are tail club guessing. For $K = \omega_1$ and $S \subseteq Seq^{\omega}(K)$ with specific origins, we have

12.1 Lemma. *Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system on A which may or may not be tail club guessing. Let us denote*
$$\mathcal{E} = \{C_\delta \lceil [m, \omega) \in Seq^{\omega}(\omega_1) \mid \delta \in A, m < \omega\}.$$
where $C_\delta \lceil [m, \omega) = \langle C_\delta(m), C_\delta(m+1), \cdots, C_\delta(k), \cdots \rangle$ and $C_\delta(k)$ denotes the k-th element of C_δ.

Then the following are equivalent.

(1) $\mathcal{E}$ is ω-stationary in $Seq^{\omega}(\omega_1)$.

(2) $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing.

Proof. (1) implies (2): Let D be a club in ω_1. We may view D as a club in $[\omega_1]^\omega$. Hence we have a map $h : [\omega_1]^{<\omega} \longrightarrow \omega_1$ such that

$$C(h) = \{a \in [\omega_1]^\omega \mid a \text{ is } h\text{-closed}\} \subseteq D.$$

Let f be the tree of clubs in $[\omega_1]^\omega$ such that for all $\langle a_0, \cdots, a_{n-1}\rangle \in {}^{<\omega}([\omega_1]^\omega)$,

$$f_{\langle a_0,\cdots,a_{n-1}\rangle} = h.$$

Then for all $\langle a_n \mid n < \omega\rangle \in B(f)$ and all $n < \omega$, we have

$$a_n \in D.$$

Since $\mathcal{E}$ is ω-stationary, we have $\langle a_n \mid n < \omega\rangle \in B(f) \cap \mathcal{E}$. Hence we have C_δ with $\delta \in A$ and $m < \omega$ such that

$$\langle a_n \mid n < \omega\rangle = C_\delta\lceil[m, \omega).$$

So

$$C_\delta \subseteq^* D.$$

(2) implies (1): Let f be a tree of clubs in $[\omega_1]^\omega$. Take a large regular cardinal θ and an $\in$-chain $\langle N_i \mid i < \omega_1\rangle$ in H_θ with $f \in N_0$. Since $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing, we have $\delta \in A$ such that

$$C_\delta \subseteq^* \{N_i \cap \omega_1 \mid i < \omega_1\}.$$

Let us reindex and may assume that there exists $m < \omega$ such that

$$C_\delta\lceil[m, \omega) = \langle N_n \cap \omega_1 \mid n < \omega\rangle.$$

Since $f_{\langle N_0\cap\omega_1,\cdots,N_{n-1}\cap\omega_1\rangle} \in N_n$, we have

$$N_n \cap \omega_1 \in C(f_{\langle N_0\cap\omega_1,\cdots,N_{n-1}\cap\omega_1\rangle}).$$

Hence

$$\langle N_n \cap \omega_1 \mid n < \omega\rangle \in B(f) \cap \mathcal{E}.$$

□

The following idea of relativizing (restricting) ω-stationary sets in $Seq^\omega(K)$ to $\mathcal{E} \uparrow K$ and considering preimages under the projection $P : Seq^\omega(K) \longrightarrow [K]^\omega$ are very important. But we sometimes directly describe related facts on these rather than using fancy terms.

12.2 Definition. (Tentative) Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system which may or may not be tail club guessing. Let $K \supseteq \omega_1$. Then we might say $S \subseteq Seq^\omega(K)$ is *$\langle C_\delta \mid \delta \in A\rangle$-$\omega$-stationary in $Seq^\omega(K)$*, if

$$S \cap (\mathcal{E} \uparrow K)$$

is ω-stationary in $Seq^\omega(K)$. We might say $S \subseteq [K]^\omega$ is *positive* in $[K]^\omega$, if

$$P^{-1}(S) = \{\langle a_n \mid n < \omega\rangle \in Seq^\omega(K) \mid \bigcup\{a_n \mid n < \omega\} \in S\}$$

is ω-stationary in $Seq^\omega(K)$. Lastly we might say $S \subseteq [K]^\omega$ is *$\langle C_\delta \mid \delta \in A\rangle$-positive* in $[K]^\omega$, if

$$P^{-1}(S) = \{\langle a_n \mid n < \omega\rangle \in Seq^\omega(K) \mid \bigcup\{a_n \mid n < \omega\} \in S\}$$

is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-stationary in $Seq^\omega(K)$.

Basic relations.

12.3 Proposition. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing and $K \supseteq \omega_1$.*

(1) If $S \subseteq Seq^\omega(K)$ is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-stationary in $Seq^\omega(K)$, then S is ω-stationary in $Seq^\omega(K)$.

(2) If $S \subseteq [K]^\omega$ is $\langle C_\delta \mid \delta \in A\rangle$-positive in $[K]^\omega$, then S is positive in $[K]^\omega$.

(3) If $S \subseteq [K]^\omega$ is positive in $[K]^\omega$, then S is stationary in $[K]^\omega$.

For $K = \omega_1$, we have a better understanding between $(TCG)^+$ of §3 and $\langle C_\delta \mid \delta \in A\rangle$-positive sets in $[K]^\omega$.

12.4 Lemma. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. Then for $S \subseteq [\omega_1]^\omega$ the following are equivalent.*

(1) S is $\langle C_\delta \mid \delta \in A\rangle$-positive in $[\omega_1]^\omega$.

(2) The preimage of $A \cap S$ under the projection

$$P : Seq^\omega(\omega_1) \longrightarrow [\omega_1]^\omega,$$

$P^{-1}(A \cap S)$, is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-stationary.

(3) $\langle C_\delta \mid \delta \in A \cap S\rangle$ is tail club guessing.

(4) $S \cap \omega_1 \in (TCG)^+$.

Proof. Similar to 12.1 lemma and playing with terminologies.

□

12.5 Corollary. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing and P be the projection*

$$P : Seq^\omega(\omega_1) \longrightarrow [\omega_1]^\omega$$

$$\langle a_n \mid n < \omega\rangle \mapsto \bigcup\{a_n \mid n < \omega\}.$$

For $X \subseteq [\omega,\ \omega_1)$, the following are equivalent.

(1) X is $\langle C_\delta \mid \delta \in A\rangle$-positive in $[\omega_1]^\omega$.

(2) $P^{-1}(X \cap A) \cap \mathcal{E}$ is ω-stationary in $Seq^\omega(\omega_1)$.

(3) $\langle C_\delta \mid \delta \in A \cap X\rangle$ is tail club guessing.

(4) $X \in (TCG)^+$, Namely, X is a positive set with respect to the tail club guessing ideal TCG.

§13. Another look at $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper preorders

We consider a characterization of our class of preorders in terms of preserving $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semistationary sets in $Seq^\omega(K)$. It is important to notice that there is a critical level $K = H_{|\mathrm{TC}(P)|^+}$ for each preorder P so that if relevant type of stationary sets are preserved there, then so are everywhere.

13.1 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing and $K \supseteq \omega_1$. We might say $S \subseteq Seq^\omega(K)$ is *ω-semistationary in $Seq^\omega(K)$*, if S^* is ω-stationary in $Seq^\omega(K)$. We might also say $S \subseteq Seq^\omega(K)$ is *$\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semistationary in $Seq^\omega(K)$*, if $(S \cap (\mathcal{E} \uparrow K))^* = S^* \cap (\mathcal{E} \uparrow K)$ is ω-stationary in $Seq^\omega(K)$, where for $T \subseteq Seq^\omega(K)$, we define

$$T^* = \{\langle b_n \mid n < \omega\rangle \in Seq^\omega(K) \mid \exists \langle a_n \mid n < \omega\rangle \in T\ \langle a_n \mid n < \omega\rangle \subseteq_{\omega_1} \langle b_n \mid n < \omega\rangle\}.$$

13.2 Note. In the above, $\langle C_\delta \mid \delta \in A\rangle$ and $\omega_1(=\bigcup A$, usually) are parameters. Even if we go bigger universes, these concepts are considered there with these parameters fixed in V. Notice that we do not assume $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing which might get lost going bigger universes. Though we are going to see that relevant objects, say, ω_1, $[K]^\omega$ and $Seq^\omega(K)$ do not change and property like $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing and so forth are preserved, these do not come free.

13.3 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. Let P be a preorder. We say P is $\langle C_\delta \mid \delta \in A\rangle$-*$\omega$-semiproper*, if for all regular cardinals λ with

$$P \in H_{|\mathrm{TC}(P)|^+} \in H_{(2^{|\mathrm{TC}(P)|})^+} \in H_\lambda$$

and all $\in$-chains $\langle N_n \mid n<\omega\rangle$ in H_λ such that $\langle N_n \cap \omega_1 \mid n<\omega\rangle \in \mathcal{E}$ and $\langle C_\delta \mid \delta \in A\rangle, P \in N_0$, if $p \in P \cap N_0$, then there exists $q \leq p$ in P such that for all $n<\omega$, q is (P, N_n)-semi-generic. Namely,

$$q \Vdash_P \text{“}N[\dot{G}] \cap \omega_1^V = N \cap \omega_1^V\text{”}.$$

where $\dot{G}$ denotes the canonical P-name for the generic objects and $N[\dot{G}] = \{\tau_{\dot{G}} \mid \tau \text{ is a } P\text{-name with } \tau \in N\}$.

If a preorder P is finite, then we have no new objects in the generic extensions. Hence infinite preorders are intended in the following.

13.4 Theorem. *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on A which may or may not be tail club guessing. Let P be a preorder with $|\,\mathrm{TC}(P)\,| \geq \omega$. Then the following are equivalent.*

(1) P is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper.

(2) For all $K \supseteq \omega_1$ and all $S \subseteq Seq^\omega(K)$ such that $S \cap (\mathcal{E} \uparrow K)$ is ω-semistationary, we have

$$\Vdash_P \text{“}(S \cap (\mathcal{E} \uparrow K))^V \text{ is } \omega\text{–semistationary in } (Seq^\omega(K))^{V[\dot{G}]}\text{”}.$$

Namely, P preseves every $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semistationary set in every $Seq^\omega(K)$ with $K \supseteq \omega_1$.

(3) (Critical level) *(2) just at $K = H_{|\mathrm{TC}(P)|^+}$.*

Proof. (1) implies (2): Let $S \subseteq Seq^\omega(K)$ such that $S \cap (\mathcal{E} \uparrow K)$ is ω-semistationary in $Seq^\omega(K)$. Suppose

$$p \Vdash_P \text{“}\dot{f} \text{ is a tree of clubs in } ([K]^\omega)^{V[\dot{G}]}\text{”}.$$

We want to find $\langle a_n \mid n<\omega\rangle \in S \cap (\mathcal{E} \uparrow K)$, $\langle \dot{b}_n \mid n<\omega\rangle$ and $q \leq p$ in P such that

$$q \Vdash_P \text{“}\langle a_n \mid n<\omega\rangle \subseteq_{\omega_1^V} \langle \dot{b}_n \mid n<\omega\rangle \in (Seq^\omega(K))^{V[\dot{G}]} \text{ such that } \langle \dot{b}_n \mid n<\omega\rangle \in (B(\dot{f}))^{V[\dot{G}]}\text{”}.$$

Let θ be a sufficiently large regular cardinal. Then we have an $\in$-chain $\langle N_n \mid n<\omega\rangle$ in H_θ and $\langle a_n \mid n<\omega\rangle \in S \cap (\mathcal{E} \uparrow K)$ such that

- $\langle C_\delta \mid \delta \in A\rangle, p, P, \dot{f} \in N_0$,
- $\langle a_n \mid n<\omega\rangle \subseteq_{\omega_1^V} \langle N_n \cap K \mid n<\omega\rangle$.
  And so
- $\langle N_n \cap \omega_1 \mid n<\omega\rangle \in \mathcal{E}$.

This is possible since $S \cap (\mathcal{E} \uparrow K)$ is ω-semistationary in $Seq^\omega(K)$. Since P is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, we have $q \leq p$ in P such that for all $n<\omega$, q is (P, N_n)-semi-generic. Hence we have

$$q \Vdash_P \text{“}\langle N_n[\dot{G}] \cap K \mid n<\omega\rangle \in (B(\dot{f}))^{V[\dot{G}]}\text{”},$$

$$q \Vdash_P \text{“}N_n[\dot{G}] \cap \omega_1^V = N_n \cap \omega_1^V = a_n \cap \omega_1^V\text{”},$$

and so

$$q \Vdash_P \text{“}\langle a_n \mid n<\omega\rangle \subseteq_{\omega_1^V} \langle N_n[\dot{G}]\cap K \mid n<\omega\rangle\text{”}.$$

Hence

$$q \Vdash_P \text{“}\langle N_n[\dot{G}]\cap K \mid n<\omega\rangle \in (((S\cap(\mathcal{E}\uparrow K))^V)^*)^{V[\dot{G}]} \cap (B(\dot{f}))^{V[\dot{G}]}\text{”}.$$

(2) implies (3): Trivial.

(3) implies (1): Let $\theta = |\mathrm{TC}(P)|^+$. Then $\theta \geq \omega_1$ is a regular cardinal such that $P \in H_\theta$. Let

$$K = H_\theta.$$

We first show that there exsists a tree of clubs f in $[K]^\omega$ such that if $\langle N_n \mid n<\omega\rangle \in B(f)\cap(\mathcal{E}\uparrow K)$, then

- $\langle N_n \mid n<\omega\rangle$ is an $\in$-chain in K,
- $P \in N_0$,
- For all $p \in P\cap N_0$, there exists $q \leq p$ such that for all $n<\omega$, q is (P, N_n)-semi-generic.

We show this by contradiction. Suppose not. Then by 10.2 lemma (Fodor's lemma), there exists $S \subseteq Seq^\omega(K)$ and $p_0 \in P$ such that $S\cap(\mathcal{E}\uparrow K)$ is ω-stationary in $Seq^\omega(K)$ and for all $\langle N_n \mid n<\omega\rangle \in S\cap(\mathcal{E}\uparrow K)$, we have

- $\langle N_n \mid n<\omega\rangle$ is an $\in$-chain in H_θ,
- $p_0, P \in N_0$,
- For all $q \leq p_0$, there exists $n<\omega$ such that q is not (P, N_n)-semi-generic.

We argue in $V[G]$ with $p_0 \in G$. Let $\dot{f}$ be a tree of clubs in $([K]^\omega)^{V[G]}$ such that

- $C(\dot{f}_{\langle \dot{a}_0,\cdots,\dot{a}_{n-1}\rangle}) \subseteq \{\dot{N} \prec K \mid \dot{N}[G]\cap K \subseteq \dot{N}\}$, where

$$C(\dot{f}_{\langle \dot{a}_0,\cdots,\dot{a}_{n-1}\rangle}) = \{\dot{N} \in ([K]^\omega)^{V[G]} \mid \dot{N} \text{ is } \dot{f}_{\langle \dot{a}_0,\cdots,\dot{a}_{n-1}\rangle}\text{–closed}\}.$$

Since $S\cap(\mathcal{E}\uparrow K)$ remains ω-semistationary, we have

$$(((S\cap\mathcal{E}\uparrow K)^V)^*)^{V[G]} \cap (B(\dot{f}))^{V[G]} \neq \emptyset.$$

Let us take $\langle N_n \mid n<\omega\rangle \in (S\cap(\mathcal{E}\uparrow K))^V$ and $\langle \dot{N}_n \mid n<\omega\rangle$ such that

$$\langle N_n \mid n<\omega\rangle \subseteq_{\omega_1^V} \langle \dot{N}_n \mid n<\omega\rangle \in (B(\dot{f}))^{V[G]}.$$

Hence

$$N_n[G]\cap\omega_1^V \subseteq \dot{N}_n[G]\cap\omega_1^V = \dot{N}_n\cap\omega_1^V = N_n\cap\omega_1^V.$$

Take a witness $q \leq p_0$ which decides $\langle N_n \mid n<\omega\rangle \in S\cap(\mathcal{E}\uparrow K)$. Hence for all $n<\omega$, we have

$$q \Vdash_P \text{“}N_n[\dot{G}]\cap\omega_1^V = N_n\cap\omega_1^V\text{”}$$

and so

$$q \text{ is } (P, N_n)\text{–semi–generic}.$$

This is a contradiction.

Claim. *Let λ be a regular cardinal with $H_\theta \in H_\lambda$. Let $\langle M_n \mid n<\omega\rangle$ be an $\in$-chain in H_λ such that $\langle C_\delta \mid \delta \in A\rangle$, $P \in M_0$ and $\langle M_n\cap\omega_1 \mid n<\omega\rangle \in \mathcal{E}$. Then for all $p \in M_0$, there exists $q \leq p$ such that for all $n<\omega$, q is (P, M_n)-semi-generic.*

Proof. Since $P \in M_0 \prec H_\lambda$, we have $H_\theta \in M_0$ and we may assume that $f \in M_0$. Hence

$$\langle M_n \cap H_\theta \mid n < \omega \rangle \in B(f) \cap (\mathcal{E} \upharpoonright H_\theta).$$

Let $p \in P \cap M_0$. Then $p \in P \cap (M_0 \cap H_\theta)$. Get $q \leq p$ such that for all $n < \omega$, q is $(P, M_n \cap H_\theta)$-semi-generic. Since $P \in H_\theta$ and $\omega_1 \leq \theta$, we have

$$q \Vdash_P \text{“} M_n[\dot{G}] \cap \omega_1^V = (M_n \cap H_\theta^V)[\dot{G}] \cap \omega_1^V = (M_n \cap H_\theta^V) \cap \omega_1^V = M_n \cap \omega_1^V \text{”}.$$

Therefore we conclude for all $n < \omega$, q is (P, M_n)-semi-generic.

□

□

§14. A quick review of iterated forcing and stages

We provide a quick review on iterated forcing and stages from [M].

14.1 Definition. We say a sequence

$$I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$$

is a *ρ-stage iteration*, if

(1) For each i, $(P_i, \leq_i)$ is a separative preorder with a greatest element 1_i and P_i consists of sequences of length i.
(2) For $p \in P_i$ and $k < i$, we have $p\lceil k = \{(\alpha, p(\alpha)) \mid \alpha < k\} \in P_k$ and $1_k = 1_i\lceil k$.
(3) For $p \in P_i$ and $a \in P_k$ with $k < i$, if $a \leq_k p\lceil k$, then we have $a \cup p\lceil[k, i) \in P_i$ and $a \cup p\lceil[k, i) \leq_i p$.
(4) For $p, q \in P_i$, if $p \leq_i q$, then for any $k < i$, we have $p\lceil k \leq_k q\lceil k$ and $p \leq_i p\lceil k \cup q\lceil[k, i)$.
(5) Let i be limit and $p, q \in P_i$. Then $p \leq_i q$ iff for all $k < i$, $p\lceil k \leq_k q\lceil k$. (order at limit)

For $p \in P_i$, we denote its length by $l(p)$ and so $l(p) = i$. The length $l(p)$ is important, since it tells which preorder p comes from. Conditions of the form $a \cup p\lceil[k, i) = a \cup \{(j, p(j)) \mid k \leq j < i\}$ are sometimes denoted by $a^\frown p\lceil[k, i)$. We abbreviate 1_i to 1 and $a^\frown 1_i\lceil[k, i)$ to $a^\frown 1$. For $p, q \in P_i$, to express $p \leq_i q$ we write either $p \leq q$ in P_i or just $p \leq q$. In these cases the value of i will be clear from the context.

We turn to intermediate stages $V[G_k]$ and the tails P_{ki} of P_i in $V[G_k]$.

14.2 Definition. Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let $k \leq i \leq \rho$. Let G_k be P_k-generic over V. Then let

$$P_{ki} = \{p\lceil[k, i) \mid p \in P_i,\ p\lceil k \in G_k\}$$

and for $x, y \in P_{ki}$, $x \leq y$ in P_{ki}, if there exists $a \in G_k$ such that $a^\frown x \leq a^\frown y$ in P_i. Then $P_k * P_{ki}$ and P_i are forcing equivalent. We call this preorder P_{ki} the *tail of P_i at k*.

For $x, y \in P_{ki}$, we write $x \equiv y$ in P_{ki}, if $x \leq y$ and $y \leq x$ in P_{ik}. This $x \equiv y$ in P_{ki} is an equivalence relation. However we do not bother to take equivalence classes, as we work with preorders.

We are interested in conditions whose contents would be exhaused in ω-many stages. Since we are dealing with iterated forcing, those stages may depend on the situation claimed by ealier stages and generic objects in use. Namely, stages come up are naturally dependent on the nature of generic objects in use. Names for stages are ready to tell how stages proceed in every situation.

14.3 Definition. Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let i be a limit ordinal with $i \leq \rho$ and $p \in P_i$. We say p has *stages* $\langle \dot{\delta}_n \mid n < \omega \rangle$, if

- $\dot{\delta}_n$'s are P_i-names,

- If $\xi \leq i$ and $x \Vdash_{P_i}$ "$\dot{\delta}_n = \check{\xi}$", then $x\lceil\xi^\frown 1 \Vdash_{P_i}$ "$\dot{\delta}_n = \check{\xi}$" (tame),
- $1 \Vdash_{P_i}$ "$\dot{\delta}_n \leq \dot{\delta}_{n+1} \leq \check{i}$" (increasing),
- $1 \Vdash_{P_i}$ "if $p\lceil\dot{\delta}_n \in \dot{G}_i\lceil\dot{\delta}_n$, then $\dot{\delta}_n < \check{i}$" for all $n < \omega$ (stage),
- $1 \Vdash_{P_i}$ "if $\dot{\delta} = \sup\{\dot{\delta}_n \mid n < \omega\}$ and $p\lceil\dot{\delta} \in \dot{G}_i\lceil\dot{\delta}$, then $p \in \dot{G}_i$", where $\dot{G}_i$ denotes the canonical P_i-name of the P_i-generic filters over V (tail).

We collect technicalities related to stages. We recap the following from [M].

14.4 Lemma. (Hooking) *Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let $i \leq \rho$ be limit. Let $y \leq x$ in P_i and $\langle \dot{\delta}^x_k \mid k < \omega \rangle$, $\langle \dot{\delta}^y_k \mid k < \omega \rangle$ be stages for x and y, respectively. Then there exist stages $\langle \tilde{\delta}^y_k \mid k < \omega \rangle$ for y such that for all $k < \omega$, we have*

$$1 \Vdash_{P_i} \text{“}\dot{\delta}^x_{k+1}, \dot{\delta}^y_k \leq \tilde{\delta}^y_k = max\{\dot{\delta}^x_{k+1}, \dot{\delta}^y_k\}\text{”}.$$

14.5 Corollary. *Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let $i \leq \rho$ be limit. Let $x \in P_i$ has stages $\langle \dot{\delta}^x_k \mid k < \omega \rangle$ and $\xi < i$.*

(1) There exist stages $\langle \tilde{\delta}^x_k \mid k < \omega \rangle$ for x such that

$$1 \Vdash_{P_i} \text{“}\check{\xi} \leq \max\{\check{\xi}, \dot{\delta}^x_k\} = \tilde{\delta}^x_k\text{”}.$$

(2) There exist stages $\langle \tilde{\delta}^x_k \mid k < \omega \rangle$ for x such that

$$1 \Vdash_{P_i} \text{“}\tilde{\delta}^x_0 = \check{\xi}\text{”}.$$

We want conditions which decide its very first stage.

14.6 Lemma. *Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let $\alpha < \alpha^* \leq \rho$ and α^* be limit. Let G_α be P_α-generic over V, $p \in P_{\alpha^*}$, $p\lceil\alpha \in G_\alpha$, $\langle \dot{\delta}_k \mid k < \omega \rangle$ stages for p and $1 \Vdash_{P_{\alpha^*}}$ "$\alpha \leq \dot{\delta}_0$". Then there exists (a, ξ) such that*

- *$\alpha \leq \xi < \alpha^*$, $a \in P_\xi$ and $a \leq p\lceil\xi$,*
- *$a\lceil\alpha \in G_\alpha$,*
- *$a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0 = \check{\xi}$",*
- *$\langle \max\{\check{\xi}, \dot{\delta}_n\} \mid n < \omega \rangle$ are stages for $a^\frown p\lceil[\xi, \alpha^*)$.*

Proof. Let $d \leq p\lceil\alpha$ in P_α.

Claim. *There exists (ξ, a) such that $\alpha \leq \xi < \alpha^*$ and $a \in P_\xi$ such that $a \leq p\lceil\xi$, $a\lceil\alpha \leq d$ and $a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0 = \check{\xi}$".*

Proof. To get these ξ and a, we may temporally take G_{α^*} which is P_{α^*}-generic over V such that $d^\frown p\lceil[\alpha, \alpha^*) \in G_{\alpha^*}$. Let $\xi = (\dot{\delta}_0)_{G_{\alpha^*}}$. Then since $p \in G_{\alpha^*}$, we have

$$\alpha \leq \xi < \alpha^*.$$

Take $q \in G_{\alpha^*}$ such that $q \leq d^\frown p\lceil[\alpha, \alpha^*)$ and $q \Vdash_{P_{\alpha^*}}$ "$\check{\xi} = \dot{\delta}_0$". Now argue in V. Since $\dot{\delta}_0$ is tame, we have $q\lceil\xi^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\check{\xi} = \dot{\delta}_0$". Let $a = q\lceil\xi$. Then $a\lceil\alpha = q\lceil\alpha \leq d$, $a \leq d^\frown p\lceil[\alpha, \xi) \leq p\lceil\xi$ and $a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0 = \check{\xi}$".

□

Let (ξ, a) be as claimed. We may assume $a\lceil\alpha \in G_\alpha$.

We next show $\langle \max\{\check{\xi}, \dot{\delta}_n\} \mid n < \omega\rangle$ are stages for $a^\frown p\lceil[\xi, \alpha^*)$.

(tame) Since $\check{\xi}$ and $\dot{\delta}_n$'s are all tame, so are $\max\{\check{\xi}, \dot{\delta}_n\}$'s.

(increasing) $\max\{\check{\xi}, \dot{\delta}_n\} \leq \max\{\check{\xi}, \dot{\delta}_{n+1}\}$.

To show (stage) and (tail), we argue in $V[\dot{G}_{\alpha^*}]$, where $\dot{G}_{\alpha^*}$ is the canonical P_{α^*}-name for the P_{α^*}-generic filters.

(stage) Let $a^\frown p\lceil[\xi, \alpha^*)\lceil\max\{\check{\xi}, \dot{\delta}_n\} \in \dot{G}_{\alpha^*}\lceil\max\{\check{\xi}, \dot{\delta}_n\}$. Then $p\lceil\max\{\check{\xi}, \dot{\delta}_n\} \in \dot{G}_{\alpha^*}\lceil\max\{\check{\xi}, \dot{\delta}_n\}$. Hence $p\lceil\dot{\delta}_n \in \dot{G}_{\alpha^*}\lceil\dot{\delta}_n$ and so $\dot{\delta}_n < \alpha^*$. Hence $\max\{\xi, \dot{\delta}_n\} < \alpha^*$.

(tail) Let $a^\frown p\lceil[\xi, \alpha^*)\lceil\sup\{\max\{\check{\xi}, \dot{\delta}_n\} \mid n < \omega\} \in \dot{G}_{\alpha^*}\lceil\sup\{\max\{\check{\xi}, \dot{\delta}_n\} \mid n < \omega\}$. Then $p\lceil\sup\{\dot{\delta}_n \mid n < \omega\} \in \dot{G}_{\alpha^*}\lceil\sup\{\dot{\delta}_n \mid n < \omega\}$. Hence $p \in \dot{G}_{\alpha^*}$ and so $a^\frown p\lceil[\xi, \alpha^*) \in \dot{G}_{\alpha^*}$.

□

14.7 Corollary. *Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho\rangle$ be an iteration. Let $\alpha < \alpha^* \leq \rho$ and α^* be limit. Let G_α be P_α-generic over V, $p \in P_{\alpha^*}$, $\langle \dot{\delta}_n \mid n < \omega\rangle$ stages for p and $p\lceil\alpha \in G_\alpha$. Then there exists $p' \leq p$ in P_{α^*} and stages $\langle \dot{\delta}'_k \mid k < \omega\rangle$ for p' such that*

- *There exist ξ with $\alpha \leq \xi < \alpha^*$ such that $p'\lceil\xi^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}'_0 = \check{\xi}$",*
- *$1 \Vdash_{P_i}$ "$\dot{\delta}_k \leq \dot{\delta}'_k$" for all $k < \omega$,*
- *$p'\lceil\alpha \in G_\alpha$.*

Proof. First consider stages $\langle \max\{\check{\alpha}, \dot{\delta}_n\} \mid n < \omega\rangle$ for p. Then apply lemma above.

□

Hence given $p \in P_{\alpha^*}$ such that p has stages $\dot{\delta}_k$ and $p\lceil\alpha \in G_\alpha$, we may assume, taking an extension if necessary, $p \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0 \leq \check{\xi}$" for some $\xi \geq \alpha$. And in this case, we actually have $p\lceil\xi^\frown(1_{\alpha^*}\lceil[\xi, \alpha^*)) \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0 \leq \check{\xi}$" by tameness.

§15. Nested antichains, fusion structures and fusions

To decide the values of a name of an ordinal, we may form an antichain. If we have stages, then we would keep deciding their values in a nested manner. Hence we formulate the following.

15.1 Definition. Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho\rangle$ be an iteration. Let i be a limit ordinal with $i \leq \rho$. We call

$$NA = (T, \langle T_n \mid n < \omega\rangle, \langle a \mapsto \mathrm{suc}^n_T(a) \mid n < \omega,\ a \in T_n\rangle)$$

is a *nested antichain in $I\lceil i$*, if

- $T = \bigcup\{T_n \mid n < \omega\}$,
- $T_0 = \{a_0\}$ for some $a_0 \in P_{l(a_0)}$ with $l(a_0) < i$ (root),
- $T_n \subseteq \bigcup\{P_k \mid k < i\}$,
- $T_{n+1} = \bigcup\{\mathrm{suc}^n_T(a) \mid a \in T_n\}$,
- For each $a \in T_n$ and $b \in \mathrm{suc}^n_T(a)$, we have $l(a) \leq l(b) < i$ and $b\lceil l(a) \leq a$ in $P_{l(a)}$,
- For each $a \in T_n$ and $b, b' \in \mathrm{suc}^n_T(a)$, if $b \neq b'$, then $b\lceil l(a)$ and $b'\lceil l(a)$ are incompatible in $P_{l(a)}$,
- For each $a \in T_n$, we have $\{b\lceil l(a) \mid b \in \mathrm{suc}^n_T(a)\}$ is a maximal antichain below a in $P_{l(a)}$,

Note that we may show $\mathrm{suc}^n_T(a) = \{b \in T_{n+1} \mid l(a) \leq l(b),\ b\lceil l(a) \leq a\}$.

We form a nested antichain by recursion and at the same time may attach a condition $p^{(n,a)}$ to each node (n, a) in the manner specified below. We know every nested antichain gives rise to a condition in the simple iterations ([M]). And such a condition would work sort of a master condition to the conditions attatched.

15.2 Definition. Let $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ be an iteration. Let i be a limit ordinal with $i \leq \rho$. Let $NA = (T, \langle T_n \mid n < \omega \rangle, \langle a \mapsto \mathrm{suc}^n_T(a) \mid n < \omega,\ a \in T_n \rangle)$ be a nested antichain in $I \lceil i$. We call

$$F = \langle (p^{(n,a)},\ \langle \dot{\delta}^{(n,a)}_k \mid k < \omega \rangle) \mid n < \omega,\ a \in T_n \rangle$$

is a *fusion structure* (on NA) in $I\lceil i$, if

- For each $a \in T_n$, we have $p^{(n,a)} \in P_i$ and $a \leq p^{(n,a)} \lceil l(a)$ in $P_{l(a)}$,
- For each $a \in T_n$ and $b \in \mathrm{suc}^n_T(a)$, we have $p^{(n+1,b)} \leq p^{(n,a)}$ in P_i,
- $\langle \dot{\delta}^{(n,a)}_k \mid k < \omega \rangle$ are stages for $p^{(n,a)}$,
- For each $a \in T_n$ and $b \in \mathrm{suc}^n_T(a)$, we have $1 \Vdash_{P_i}$ "$\dot{\delta}^{(n,a)}_{k+1} \leq \dot{\delta}^{(n+1,b)}_k$" (A step ahead),
- $p^{(n,a)}\lceil l(a)^\frown 1 \Vdash_{P_i}$ "$\dot{\delta}^{(n,a)}_0 = \check{l(a)}$". (This is sufficient to have $a^\frown 1 \Vdash_{P_i}$ "$\dot{\delta}^{(n,a)}_0 = \check{l(a)}$".)

  We call $p \in P_i$ is a *fusion* of the fusion structure F, if

- $1 \Vdash_{P_i}$ "$p \in \dot{G}_i$ iff there exists a sequence $\langle \dot{a}_n \mid n < \omega \rangle$ such that for all $n < \omega$, we have $\dot{a}_n \in T_n \cap \dot{G}_i \lceil l(\dot{a}_n)$, $\dot{a}_{n+1} \in \mathrm{suc}^n_T(\dot{a}_n)$ and so in this case, for all $n < \omega$, we have $p^{(n,\dot{a}_n)} \in \dot{G}_i$".

  We refer to $\langle \dot{a}_n \mid n < \omega \rangle$ as a *generic cofinal path through* T.

§16. Simple iterations of semiproper preorders

We introduce our revised countable support iterations of [M]. We refer to them as *simple* iterations. They satisfy properties listed below. To construct a simple iteration we specify how we force at each successor stage. Namely, $\dot{Q}_i$. We take what [M] calls the *simple limit* at each limit stage. The simple limit is a suborder of the inverse limit. Each condition in the limit has its stages. Each fusion structure in the limit has its fusion in the limit. For details of simple iterations, see [M], where $\dot{Q}_i$'s are dealt implicitly.

16.1 Lemma. *If* $SI = (\langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle, \langle (\dot{Q}_i, \dot{\leq}_i, \dot{1}_i) \mid i < \rho \rangle)$ *is a simple iteration, then*

(1) $I = \langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle$ *is an iteration,*

(2) $1 \Vdash_{P_i}$ "$(\dot{Q}_i, \dot{\leq}_i, \dot{1}_i)$ *is a separative preorder",*

(3) P_{i+1} *and* $P_i * \dot{Q}_i$ *are forcing equivalent and* $1_{i+1} = 1_i \cup \{(i, \dot{1}_i)\}$,

(4) For limit i, *if* $p \in P_i$, *then* p *has some stages* $\langle \dot{\delta}_n \mid n < \omega \rangle$,

(5) For limit i, *if* $F = \langle (p^{(n,a)},\ \langle \dot{\delta}^{(n,a)}_k \mid k < \omega \rangle) \mid a \in T_n,\ n < \omega \rangle$ *is a fusion structure in* $I\lceil i$, *then there exists a fusion* $p \in P_i$ *of* F,

(6) For limit i, *if* $p \in P_i$, *then there exists a fusion structure* F *in* $I\lceil i$ *such that if* $q \in P_i$ *is a fusion of* F, *then* $q \leq p$ *in* P_i,

(7) If $k < i$, τ *is a* P_k*-name and* $a \Vdash_{P_k}$ "$\tau \in P_i$ *and* $\tau\lceil k \in \dot{G}_k$", *then there exists* $q \in P_i$ *such that* $q\lceil k = a$ *and* $a \Vdash_{P_k}$ "$\tau\lceil[k,i) \equiv q\lceil[k,i)$ *in* P_{ki}" (fullness).

16.2 Definition. Let $SI = (\langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle, \langle (\dot{Q}_i, \dot{\leq}_i, \dot{1}_i) \mid i < \rho \rangle)$ be a simple iteration. For $p \in P_i$, the *support* of p is defined by

$$\mathrm{support}(p) = \{k < i \mid p(k) \neq \dot{1}_k\}.$$

support(p) may or may not be countable.

We recap from [M] the iteration lemma for semiproperness under simple iterations.

16.3 Lemma. (Iteration lemma for semiproper) *Let* $SI = (\langle (P_i, \leq_i, 1_i) \mid i \leq \rho \rangle, \langle (\dot{Q}_i, \dot{\leq}_i, \dot{1}_i) \mid i < \rho \rangle)$ *be a simple iteration such that for all* $i < \rho$ *we assume*

$$1 \Vdash_{P_i} \text{“}\dot{Q}_i \textit{ are semiproper”}.$$

Let θ be a sufficiently large regular cardinal and $SI \in N \prec H_\theta$. These θ and N are fixed once for all. Then we have:

<u>IF</u> $(a, i, i^*, p, \dot{M})$ *satisfy*

(1) $i \le i^* \le \rho$, $a \in P_i$, $p \in P_{i^*}$ *and* $a \le p\lceil i$ *in* P_i,

(2) $a \Vdash_{P_i}$ "$N \cup \{\dot{G}_i, p\} \subseteq \dot{M} \prec H_\theta^{V[\dot{G}_i]}$",

<u>THEN</u> *there exists* $a^* \in P_{i^*}$ *such that*

(3) $a^*\lceil i = a$ *and* $a^* \le p$ *in* P_{i^*},

(4) $a \Vdash_{P_i}$ "$a^*\lceil[i, i^*)$ *is* $(P_{ii^*}, \dot{M})$*-semi-generic*".

Hence we have the following in $V[G_i]$ *for all* G_i *with* $i \le \rho$:

If $N \cup \{G_i\} \subseteq M \prec H_\theta^{V[G_i]}$, $i \le i^* \le \rho$, $x \in P_{i^*} \cap M$ *and* $x\lceil i \in G_i$, *then there exists* $y \le x$ *in* P_{i^*} *such that* $y\lceil i \in G_i$ *and* $y\lceil[i, i^*)$ *is* (P_{ii^*}, M)*-semi-generic.*

Lastly we recall things related to chain conditions of simple iterations from [M].

16.4 Lemma. *Let κ be a regular uncountable cardinal. Let* $SI = (\langle (P_i, \le_i, 1_i) \mid i \le \kappa\rangle, \langle(\dot{Q}_i, \dot{\le}_i, \dot{1}_i) \mid i < \kappa\rangle)$ *be a simple iteration.*

(1) If for all $i < \kappa$, we have $|P_i| < \kappa$, then every condition in P_κ is extended to a condition with bounded support in P_κ.

(2) If κ is Mahlo and for all $i < \kappa$, we have $|P_i| < \kappa$, then P_κ has the κ-c.c.

We are not sure about the size of P_κ other than $|P_\kappa| \le \Pi_{\alpha<\kappa}|P_\alpha|$. However, it is very much close to the direct limit in some cases.

16.5 Proposition. *Let κ be a regular uncountable cardinal. Let $I = \langle (P_i, \le_i, 1_i) \mid i \le \kappa\rangle$ be an iteration. If P_κ has the κ-c.c and the conditions with bounded supports are dense in P_κ, then for every condition $p \in P_\kappa$, there exists $\xi < \kappa$ with $p \equiv p\lceil\xi^\frown 1$.*

Proof. Let $p \in P_\kappa$. Take a maximal antichain $\mathcal{A}$ below p. We may assume every member of $\mathcal{A}$ has bounded support. Since P_κ has the κ-c.c, there exists $\xi < \kappa$ such that for all $q \in \mathcal{A}$, support$(q) \subset \xi$. We show

$$p \equiv p\lceil\xi^\frown 1.$$

Let G_κ be P_κ-generic over V with $p\lceil\xi^\frown 1 \in G_\kappa$. Let G'_κ be P_κ-generic over V such that $G'_\kappa\lceil\xi = G_\kappa\lceil\xi$ and $p \in G'_\kappa$. Then there exists $q \in \mathcal{A} \cap G'_\kappa$. Since $q\lceil\xi \in G'_\kappa\lceil\xi$, we have $q\lceil\xi \in G_\kappa\lceil\xi$. Since $q = q\lceil\xi^\frown 1$, we conclude $q \in G_\kappa$. Since $q \le p$, we have $p \in G_\kappa$. Since P_κ is separative, we are done.

□

§17. Iteration lemma and theorem

Now we are ready to show the following.

17.1 Lemma. (Iteration lemma for semiproper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper) *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on $A \subseteq \{\alpha < \omega_1 \mid \alpha$ is limit $\}$ which may or may not be tail club guessing. Let us denote*

$$\mathcal{E} = \bigcup\{C_\delta\lceil[m, \omega) \mid m < \omega,\ \delta \in A\} \subset Seq^\omega(\omega_1).$$

Let

$$SI = (\langle (P_i, \le_i, 1_i) \mid i \le \rho\rangle, \langle(\dot{Q}_i, \dot{\le}_i, \dot{1}_i) \mid i < \rho\rangle)$$

be a simple iteration such that for all $i < \rho$,

- $\Vdash_{P_i}$ "$\dot{Q}_i$ *are semiproper*",
- $\Vdash_{P_i}$ "$\dot{Q}_i$ *are* $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-*semiproper*".

Then for all sufficiently large regular cardinals θ *and all* $N \prec H_\theta$ *with* $\langle C_\delta \mid \delta \in A\rangle, SI \in N$, *we have the following:*

IF $(w, \alpha, \alpha^*, p, \langle \dot{M}_n \mid n < \omega\rangle)$ *satisfies*

(1) $\alpha \le \alpha^* \le \rho$, $w \in P_\alpha$, $p \in P_{\alpha^*}$ *and* $w \le p\lceil\alpha$ *in* P_α,

(2) $w \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega\rangle$ *is an* $\in$-*chain in* $H_\theta^{V[\dot{G}_\alpha]}$",

(3) $w \Vdash_{P_\alpha}$ "$\langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$".

THEN *there exists* $w^* \in P_{\alpha^*}$ *such that*

(4) $w^*\lceil\alpha = w$ *and* $w^* \le p$ *in* P_{α^*},

(5) $w \Vdash_{P_\alpha}$ "*for all* $n < \omega$, $w^*\lceil[\alpha, \alpha^*)$ *is* $(P_{\alpha\alpha^*}, \dot{M}_n)$-*semi-generic*".

Therefore in V^{P_α}, *the following holds.*

Let us assume that

- $N \cup \{\dot{G}_\alpha\} \subseteq \dot{M}_0 \prec H_\theta^{V[\dot{G}_\alpha]}$,
- $\langle \dot{M}_n \mid n < \omega\rangle$ *is an* $\in$-*chain in* $H_\theta^{V[\dot{G}_\alpha]}$,
- $\langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$,
- $x \in P_{\alpha^*} \cap \dot{M}_0$ *and* $x\lceil\alpha \in \dot{G}_\alpha$.

Then there exists y *such that*

- $y \le x$ *in* P_{α^*},
- $y\lceil\alpha \in \dot{G}_\alpha$,
- *For all* $n < \omega$, *we have* $y\lceil[\alpha, \alpha^*)$ *is* $(P_{\alpha\alpha^*}, \dot{M}_n)$-*semi-generic.*

Proof. θ and N are fixed once for all. We proceed by induction on α^*.

Notation. If $\dot{M}$ is a P_α-name with $w \Vdash_{P_\alpha}$ "$\dot{M} \prec H_\theta^{V[\dot{G}_\alpha]}$" and G_α is P_α-generic over V with $w \in G_\alpha$, then $\dot{M}_{G_\alpha}$ denotes the interpretation of $\dot{M}$ by G_α in $V[G_\alpha]$. If $\alpha \le \beta$ and $P_{\alpha\beta} \in \dot{M}_{G_\alpha}$ and $G_{\alpha\beta}$ is $P_{\alpha\beta}$-generic over $V[G_\alpha]$, then $\dot{M}[G_{\alpha\beta}]$ abbreviates

$$\dot{M}_{G_\alpha}[G_{\alpha\beta}] = \{\tau_{G_{\alpha\beta}} \mid \tau \in \dot{M}_{G_\alpha} \cap (V[G_\alpha])^{P_{\alpha\beta}}\} \prec H_\theta^{V[G_\beta]} = H_\theta^{V[G_\alpha][G_{\alpha\beta}]} = (H_\theta)^{V[G_\alpha]}[G_{\alpha\beta}],$$

where

$$G_\beta = G_\alpha * G_{\alpha\beta}, \quad (H_\theta)^{V[G_\alpha]}[G_{\alpha\beta}] = \{\tau_{G_{\alpha\beta}} \mid \tau \in H_\theta^{V[G_\alpha]} \cap (V[G_\alpha])^{P_{\alpha\beta}}\}.$$

If $\alpha \le \beta$ and G_β is P_β-generic over V, then for P_α-names $\dot{M}$, it would not be precise to write $\dot{M}_{G_\beta}$. We denote it by $\dot{M}_{G_\beta\lceil\alpha}$, as $G_\beta\lceil\alpha = \{y\lceil\alpha \mid y \in G_\beta\}$ is P_α-generic over V.

Case. $(\beta \longrightarrow \beta + 1)$:

Let $(w, \beta, \beta + 1, p, \langle \dot{M}_n \mid n < \omega\rangle)$ satisfy

(1) $\beta < \beta + 1 \le \rho$, $w \in P_\beta$, $p \in P_{\beta+1}$ and $w \le p\lceil\beta$,

(2) $w \Vdash_{P_\beta}$ "$N \cup \{\dot{G}_\beta, p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\beta]}$",

(3) $w \Vdash_{P_\beta}$ "$\langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$".

In $V[G_\beta]$ with $w \in G_\beta$, let $M_n = (\dot{M}_n)_{G_\beta}$. We assume $Q_\beta = (\dot{Q}_\beta)_{G_\beta}$ is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper and we now have

- $\langle C_\delta \mid \delta \in A\rangle, Q_\beta \in M_0$, $\langle M_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[G_\beta]}$,
- $(p(\beta))_{G_\beta} \in Q_\beta \cap M_0$,
- $\langle M_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$.

Hence we get $(w(\beta))_{G_\beta} \in Q_\beta$ such that

- For all $n < \omega$, $(w(\beta))_{G_\beta}$ is (Q_β, M_n)-semi-generic,
- $(w(\beta))_{G_\beta} \leq (p(\beta))_{G_\beta}$.

Let $w^* = w^\frown\langle w(\beta)\rangle$ in V. Then this w^* works.

Case. $(\alpha \longrightarrow \beta \longrightarrow \beta+1)$:

Let $\big(w, \alpha, \beta+1, p, \langle \dot{M}_n \mid n < \omega\rangle\big)$ satisfy

(1) $\alpha < \beta < \beta+1 \leq \rho$, $w \in P_\alpha$, $p \in P_{\beta+1}$ and $w \leq p\lceil\alpha$,

(2) $w \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\alpha]}$",

(3) $w \Vdash_{P_\alpha}$ "$\langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$".

Hence $\big(w, \alpha, \beta, p\lceil\beta, \langle \dot{M}_n \mid n < \omega\rangle\big)$ satisfy

(1) $\alpha < \beta < \rho$, $w \in P_\alpha$, $p\lceil\beta \in P_\beta$ and $w \leq (p\lceil\beta)\lceil\alpha$,

(2) $w \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, \underline{p\lceil\beta}\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\alpha]}$",

(3) $w \Vdash_{P_\alpha}$ "$\langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle \in \mathcal{E}$".

Apply induction hypothesis at β. We get $w^* \in P_\beta$ such that

(4) $w^*\lceil\alpha = w$ and $w^* \leq p\lceil\beta$,

(5) $w \Vdash_{P_\alpha}$ "for all $n < \omega$, $w^*\lceil[\alpha, \beta)$ is $(P_{\alpha\beta}, \dot{M}_n)$-semi-generic".

Then $\big(w^*, \beta, \beta+1, p, \underline{\langle \dot{M}_n[\dot{G}_{\alpha\beta}] \mid n < \omega\rangle}\big)$ satisfy

(1) $\beta < \beta+1 \leq \rho$, $w^* \in P_\beta$, $p \in P_{\beta+1}$ and $w^* \leq p\lceil\beta$,

(2) $w^* \Vdash_{P_\beta}$ "$N \cup \underline{\{\dot{G}_\beta, p\} \subseteq \dot{M}_0[\dot{G}_{\alpha\beta}]}$, $\underline{\langle \dot{M}_n[\dot{G}_{\alpha\beta}] \mid n < \omega\rangle}$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\beta]}$",

(3) $w^* \Vdash_{P_\beta}$ "$\underline{\langle \dot{M}_n[\dot{G}_{\alpha\beta}] \cap \omega_1^V \mid n < \omega\rangle = \langle \dot{M}_n \cap \omega_1^V \mid n < \omega\rangle} \in \mathcal{E}$".

Proof. Argue in $V[G_\beta]$ with $w^* \in G_\beta$. Let $G_\alpha = G_\beta\lceil\alpha$, $G_{\alpha\beta} = G_\beta\lceil[\alpha, \beta)$ and $(\dot{M}_n)_{G_\alpha} = M_n$.

Claim 1. $\{G_\beta, p\} \subseteq M_0[G_{\alpha\beta}]$.

Proof. $G_\alpha, P_\beta \in M_0$ and so $P_{\alpha\beta} \in M_0$. So $M_0[G_{\alpha\beta}] \prec H_\theta^{V[G_\beta]}$. We have $G_\beta = G_\alpha * G_{\alpha\beta} \in M_0[G_{\alpha\beta}]$.

Claim 2. *$\langle M_n[G_{\alpha\beta}] \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[G_\beta]}$.*

Proof. $M_n \in M_{n+1}$, $G_{\alpha\beta} \in M_{n+1}[G_{\alpha\beta}]$. Hence

$$M_n[G_{\alpha\beta}] = \{\tau_{G_{\alpha\beta}} \mid \tau \in M_n \cap (V[G_\alpha])^{P_{\alpha\beta}}\} \in M_{n+1}[G_{\alpha\beta}] \prec H_\theta^{V[G_\beta]}.$$

Claim 3. $M_n[G_{\alpha\beta}] \cap \omega_1^V = M_n \cap \omega_1^V$.

Proof. $w^*\lceil[\alpha, \beta) \in G_{\alpha\beta}$ and is $(P_{\alpha\beta}, M_n)$-semi-generic. So $M_n[G_{\alpha\beta}] \cap \omega_1^V = M_n \cap \omega_1^V$.

□

Now by case $(\beta \longrightarrow \beta+1)$, get $w^* \in P_{\beta+1}$ such that

(4') $w^*\lceil\beta = w^*$ and $w^* \leq p$ in $P_{\beta+1}$, (5') $w^* \Vdash_{P_\beta}$ " for all $n < \omega$, $w^*\lceil[\beta, \beta+1)$ is $(P_{\beta\beta+1}, \dot{M}_n[\dot{G}_{\alpha\beta}])$-semi-generic".

Hence

(4) $w^*\lceil\alpha = w^*\lceil\alpha = w$ and $w^* \leq p$ in $P_{\beta+1}$,

(5) $w \Vdash_{P_\alpha}$ "for all $n < \omega$, $w^*\lceil[\alpha, \beta+1)$ <u>is</u> $(P_{\alpha\beta+1}, \dot{M}_n)$-semi-generic".

Proof. Argue in $V[G_{\beta+1}]$ with $w^* \in G_{\beta+1}$. Let G_α, $G_{\alpha\beta}$ and M_n's be as indicated.

$$M_n[G_{\alpha\beta}] \cap \omega_1^V = M_n \cap \omega_1^V,$$

as $w^*\lceil[\alpha, \beta) \in G_{\alpha\beta}$.

$$M_n[G_{\alpha\beta}][G_{\beta\beta+1}] \cap \omega_1^V = M_n[G_{\alpha\beta}] \cap \omega_1^V,$$

as $w^*\lceil[\beta, \beta+1) \in G_{\beta\beta+1}$. Hence

$$M_n[G_{\alpha\beta+1}] \cap \omega_1^V = M_n[G_{\alpha\beta}][G_{\beta\beta+1}] \cap \omega_1^V = M_n \cap \omega_1^V.$$

□

Case. Limit(α^*):

Let $(w, \alpha, \alpha^*, p, \langle \dot{M}_n \mid n < \omega \rangle)$ satisfy

(1) $\alpha < \alpha^* \leq \rho$, $w \in P_\alpha$, $p \in P_{\alpha^*}$ and $w \leq p\lceil\alpha$,

(2) $w \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega \rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\alpha]}$",

(3) $w \Vdash_{P_\alpha}$ "$\langle \dot{M}_n \cap \omega_1^V \mid n < \omega \rangle \in \mathcal{E}$".

Construct by recursion on $k < \omega$,

$$\langle (k, a) \mapsto (p^{(k,a)}, \langle \dot{\delta}_l^{(k,a)} \mid l < \omega \rangle, \langle \dot{M}_n^{(k,a)} \mid n < \omega \rangle) \mid k < \omega,\ a \in T_k \rangle$$

such that

<u>$T_0 = \{a_0\}$</u>:

Let $T_0 = \{w\}$, $p^{(0,w)} = p$, $\langle \dot{\delta}_l^{(0,w)} \mid l < \omega \rangle$ be stages for $p^{(0,w)}$ with $\dot{\delta}_0^{(0,w)} = \check{\alpha}$ and $\dot{M}_n^{(0,w)} = \dot{M}_n$. We may assume $w \Vdash_{P_\alpha}$ "$\langle \dot{\delta}_l^{(0,w)} \mid l < \omega \rangle \in \dot{M}_0$".

<u>$T_k \longrightarrow T_{k+1}$</u>:

Suppose we have constructed

$$\langle (k, a) \mapsto (p^{(k,a)}, \langle \dot{\delta}_l^{(k,a)} \mid l < \omega \rangle, \langle \dot{M}_n^{(k,a)} \mid n < \omega \rangle) \mid a \in T_k \rangle$$

such that for each $a \in T_k$, we have

- $a \leq p^{(k,a)}\lceil l(a)$ and $p^{(k,a)} \leq p$ in P_{α^*},
- $p^{(k,a)}$ has stages $\langle \dot{\delta}_l^{(k,a)} \mid l < \omega \rangle$,
- $p^{(k,a)}\lceil l(a)^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0^{(k,a)} = l(a)$",

(2) $a \Vdash_{P_{l(a)}}$ "$N \cup \{\dot{G}_{l(a)}, p^{(k,a)}, \langle \dot{\delta}_l^{(k,a)} \mid l < \omega \rangle\} \subseteq \dot{M}_0^{(k,a)}$, $\langle \dot{M}_n^{(k,a)} \mid n < \omega \rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_{l(a)}]}$",

(3) $a \Vdash_{P_{l(a)}}$ "$\langle \dot{M}_n^{(k,a)} \cap \omega_1^V \mid n < \omega \rangle \in \mathcal{E}$".

Apply 16.3 lemma (iteration lemma for semiproper) to $(a, l(a), \alpha^*, p^{(k,a)}, \dot{M}_0^{(k,a)})$ which satisfies

- $l(a) < \alpha^*$, $a \in P_{l(a)}$, $p^{(k,a)} \in P_{\alpha^*}$ and $a \leq p^{(k,a)}\lceil l(a)$,

- $a \Vdash_{P_{l(a)}}$ "$N \cup \{\dot{G}_{l(a)}, p^{(k,a)}\} \subseteq \dot{M}_0^{(k,a)} \prec H_\theta^{V[\dot{G}_{l(a)}]}$".

Get a $P_{l(a)}$-name $\dot{p}$ such that a forces the following in $V^{P_{l(a)}}$ via 14.7 corollary.

- $\dot{p} \leq p^{(k,a)}$ in P_{α^*},
- $\dot{p}\lceil l(a) \in \dot{G}_{l(a)}$,
- $\dot{p}\lceil[l(a), \alpha^*)$ is $(P_{l(a)\alpha^*}, \dot{M}_0^{(k,a)})$-semi-generic,
- $\langle \dot{\delta}_l^{\dot{p}} \mid l < \omega\rangle$ are stages for $\dot{p}$ and for all $l < \omega$, $1 \Vdash^V_{P_{\alpha^*}}$ "$\dot{\delta}_{l+1}^{(k,a)} \leq \dot{\delta}_l^{\dot{p}}$" (a step ahead),
- There is α' such that $l(a) \leq \alpha'$, $\dot{p}\lceil\alpha'^\frown 1 \Vdash^V_{P_{\alpha^*}}$ "$\dot{\delta}_0^{\dot{p}} = \check{\alpha}'$",
- $\alpha', \dot{p}, \langle \dot{\delta}_l^{\dot{p}} \mid l < \omega\rangle \in \dot{M}_1^{(k,a)}$.

By considering d's which decide the values of $\dot{p}$ and α', we have a map $\langle d \mapsto (p_d, \langle \dot{\delta}_l^{p_d} \mid l < \omega\rangle, \beta_d)\rangle$ whose domain is dense below a in $P_{l(a)}$. Hence for each d in the domain, we have

- $d \leq a$ in $P_{l(a)}$,
- $d \leq p_d\lceil l(a)$ in $P_{l(a)}$ and $p_d \leq p^{(k,a)}$ in P_{α^*},
- $d \Vdash_{P_{l(a)}}$ "$p_d\lceil[l(a), \alpha^*)$ is $(P_{l(a)\alpha^*}, \dot{M}_0^{(k,a)})$-semi-generic"
- p_d has stages $\langle \dot{\delta}_l^{p_d} \mid l < \omega\rangle$ such that $1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_{l+1}^{(k,a)} \leq \dot{\delta}_l^{p_d}$" (a step ahead),
- $l(a) \leq \beta_d < \alpha^*$,
- $p_d\lceil\beta_d^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0^{p_d} = \check{\beta}_d$".
- $d \Vdash_{P_{l(a)}}$ "$\beta_d, p_d, \langle \dot{\delta}_l^{p_d} \mid l < \omega\rangle \in \dot{M}_1^{(k,a)}$".

Now apply induction hypothesis to $(d, l(a), \beta_d, p_d\lceil\beta_d, \underline{\langle M_1^{(k,a)}, M_2^{(k,a)}, \cdots\rangle})$ which satisfies

(1) $l(a) \leq \beta_d < \alpha^*$, $d \in P_{l(a)}$, $p_d\lceil\beta_d \in P_{\beta_d}$ and $d \leq (p_d\lceil\beta_d)\lceil l(d)$,

(2) $d \Vdash_{P_{l(a)}}$ "$N \cup \{\dot{G}_{l(a)}, p_d\lceil\beta_d\} \subseteq \dot{M}_1^{(a,k)}$ and $\langle \dot{M}_1^{(k,a)}, \dot{M}_2^{(k,a)}, \cdots\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_{l(a)}]}$",

(3) $d \Vdash_{P_{l(a)}}$ "$\langle \dot{M}_1^{(k,a)} \cap \omega_1^V, \dot{M}_2^{(k,a)} \cap \omega_1^V, \cdots\rangle \in \mathcal{E}$".

Get $d^* \in P_{\beta_d}$ such that

- $d^*\lceil l(a) = d$ and $d^* \leq p_d\lceil\beta_d$,
- $d \Vdash_{P_{l(a)}}$ " for all $n = 1, 2, \cdots$, we have $d^*\lceil[l(a), \beta_d)$ is $(P_{l(a)\beta_d}, \dot{M}_n^{(k,a)})$-semi-generic".

Fix $\mathrm{suc}_T^k(a)$ among the d^* and a map

$$\langle b \mapsto (p^{(k+1,b)}, \langle \dot{\delta}_l^{(k+1,b)} \mid l < \omega\rangle, \langle \dot{M}_n^{(k+1,b)} \mid n < \omega\rangle) \mid b \in \mathrm{suc}_T^k(a)\rangle$$

so that

- $b \leq p^{(k+1,b)}\lceil l(b)$ and $p^{(k+1,b)} \leq p^{(k,a)} \leq p$ in P_{α^*},
- $p^{(k+1,b)}$ has stages $\langle \dot{\delta}_l^{(k+1,b)} \mid l < \omega\rangle$,
- $p^{(k+1,b)}\lceil l(b)^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0^{(k+1,b)} = l(b)$",
- $1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_{l+1}^{(k,a)} \leq \dot{\delta}_l^{(k+1,b)}$" (a step ahead),
- $b\lceil l(a) \Vdash_{P_{l(a)}}$ "$l(b), p^{(k+1,b)}, \langle \dot{\delta}_l^{(k+1,b)} \mid l < \omega\rangle \in \dot{M}_1^{(k,a)}$ and so $P_{l(a)l(b)} \in \dot{M}_1^{(k,a)}$",
- $b\lceil l(a) \Vdash_{P_{l(a)}}$ "$p^{(k+1,b)}\lceil[l(a), \alpha^*)$ is $(P_{l(a)\alpha^*}, \dot{M}_0^{(k,a)})$-semi-generic",
- $b\lceil l(a) \Vdash_{P_{l(a)}}$ "for all $n = 1, 2, \cdots$, we have $b\lceil[l(a), l(b))$ is $(P_{l(a)l(b)}, \dot{M}_n^{(k,a)})$-semi-generic",
- $b \Vdash_{P_{l(b)}}$ "$\dot{M}_n^{(k+1,b)} = \dot{M}_{n+1}^{(k,a)}[\dot{G}_{l(a)l(b)}]$ are well-defined for $n < \omega$",

(2) $b \Vdash_{P_{l(b)}}$ "$N \cup \{\dot{G}_{l(b)}, p^{(k+1,b)}\} \subseteq \dot{M}_1^{(k,a)}[G_{l(a)l(b)}] = M_0^{(k+1,b)}$ and $\langle \dot{M}_n^{(k+1,b)} \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_{l(b)}]}$",

(3) $b \Vdash_{P_{l(b)}}$ "$\langle \dot{M}_n^{(k+1,b)} \cap \omega_1^V \mid n < \omega\rangle = \langle \dot{M}_1^{(k,a)} \cap \omega_1^V, \dot{M}_2^{(k,a)} \cap \omega_1^V, \cdots\rangle \in \mathcal{E}$".

This completes the construction.

Let q be a fusion of the fusion structure F in $I\lceil\alpha^*$, where

$$F = \langle (k,a) \mapsto (p^{(k,a)}, \langle \dot{\delta}_l^{(k,a)} \mid l < \omega\rangle) \mid k < \omega,\ a \in T_k\rangle.$$

Namely, we have

- $p^{(k,a)} \in P_{\alpha^*}$ and $a \leq p^{(k,a)}\lceil l(a)$,
- If $b \in \mathrm{suc}_T^k(a)$, then $p^{(k+1,b)} \leq p^{(k,a)}$ in P_{α^*},
- $\langle \dot{\delta}_l^{(k,a)} \mid l < \omega\rangle$ are stages for $p^{(k,a)} \in P_{\alpha^*}$,
- $\langle \dot{\delta}_l^{(k+1,b)} \mid l < \omega\rangle$ is a step ahead of $\langle \dot{\delta}_l^{(k,a)} \mid l < \omega\rangle$,
- $p^{(k,a)}\lceil l(a)^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0^{(k,a)} = l(a)$" and so $a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_0^{(k,a)} = l(a)$",
- $q \Vdash_{P_{\alpha^*}}$ "there exists a generic cofinal path $\langle \dot{a}_k \mid k < \omega\rangle$ through T such that $p^{(k,\dot{a}_k)} \in \dot{G}_{\alpha^*}$".

Argue in $V[G_{\alpha^*}]$ with $q \in G_{\alpha^*}$, let

$$a_k = (\dot{a}_k)_{G_{\alpha^*}},\ \alpha_k = l(a_k),\ G_{\alpha_k} = G_{\alpha^*}\lceil\alpha_k,\ p_k = p^{(a_k,k)},\ M_n^k = (\dot{M}_n^{(k,a_k)})_{G_{\alpha_k}},$$

$$M_n = (\dot{M}_n)_{G_\alpha}.$$

Claim. *For all $n < \omega$, we have*

$$M_n[G_{\alpha\alpha^*}] \cap \omega_1^V = M_n \cap \omega_1^V.$$

And this shows $q \leq p$ in P_{α^*}, $w \Vdash_{P_\alpha}$ "for all $n < \omega$, we have $q\lceil[\alpha,\alpha^*)$ is $(P_{\alpha\alpha^*}, \dot{M}_n)$-semi-generic". Since we may assume $q\lceil\alpha = w$, we are done.

Proof. We have

$$M_1[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha^*}] \supseteq M_1[G_{\alpha_0\alpha^*}],$$
$$M_2[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^*}] \supseteq M_2[G_{\alpha_0\alpha^*}],$$
$$\vdots$$
$$M_n[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}]\cdots[G_{\alpha_{n-1}\alpha_n}][G_{\alpha_n\alpha^*}] \supseteq M_n[G_{\alpha_0\alpha^*}],$$
$$\vdots$$

And so

$$M_0 \cap \omega_1^V = M_0[G_{\alpha_0\alpha^*}] \cap \omega_1^V,$$
$$M_1 \cap \omega_1^V = M_1[G_{\alpha_0\alpha_1}] \cap \omega_1^V = M_1[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha^*}] \cap \omega_1^V = M_1[G_{\alpha_0\alpha^*}] \cap \omega_1^V,$$
$$M_2 \cap \omega_1^V = M_2[G_{\alpha_0\alpha_1}] \cap \omega_1^V = M_2[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}] \cap \omega_1^V = M_2[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^*}] \cap \omega_1^V = M_2[G_{\alpha_0\alpha^*}] \cap \omega_1^V,$$
$$\vdots$$

This way we conclude

$$M_n[G_{\alpha\alpha^*}] \cap \omega_1^V = M_n \cap \omega_1^V.$$

□

17.2 Theorem. (Iteration theorem for semiproper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper) *Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system on $A \subseteq \{\alpha < \omega_1 \mid \alpha \text{ is limit}\}$ which may or may not be tail club guessing. Let*

$$SI = \big(\langle (P_i, \leq_i, 1_i) \mid i \leq \rho\rangle, \langle (\dot{Q}_i, \dot{\leq}_i, \dot{1}_i) \mid i < \rho\rangle\big)$$

be a simple iteration such that for all $i < \rho$ we assume

$$1 \Vdash_{P_i} \text{“}\dot{Q}_i \text{ are semiproper”}.$$

$$1 \Vdash_{P_i} \text{“}\dot{Q}_i \text{ are } \langle C_\delta \mid \delta \in A\rangle\text{–}\omega\text{–semiproper”}.$$

Then for all sufficiently large regular cardinals θ and all $\in$-chains $\langle N_n \mid n < \omega\rangle$ in H_θ such that

$$\langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{E}$$

and $\langle C_\delta \mid \delta \in A\rangle, SI \in N_0$, if $p \in P_\rho \cap N_0$, then there exists $q \leq p$ in P_ρ such that for all $n < \omega$, q is (P_ρ, N_n)-semi-generic.

Hence P_ρ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper.

Proof. Let $N = N_0$, $\alpha = 0$, $\alpha^* = \rho$, $w = \emptyset$, $\dot{M}_n = \check{N}_n$ in $P_\emptyset$. Get $w^* \in P_\rho$ such that $w^* \leq p$ and for all $n < \omega$, w^* is (P_ρ, N_n)-semi-generic.

□

§18. A forcing axiom compatible with tail club guessing

18.1 Theorem. *Let κ be a supercompact cardinal and $\langle C_\delta \mid \delta \in A\rangle$ be tail club guessing. Then there exists a notion of forcing P such that P is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper and in the generic extensions $W = V^P$, we have*

(1) $\langle C_\delta \mid \delta \in A\rangle$ remains tail club guessing,

(2) The $+$-type forcing axiom holds for all preorders which are semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper. Namely, if Q is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, $\langle D_i \mid i < \omega_1\rangle$ is a sequence of dense subsets of Q and $\dot{S}$ is a Q-name of a stationary subset of ω_1 in W^Q, then there exists a filter F in Q such that for all $i < \omega_1$, $D_i \cap F \neq \emptyset$ and $\{\alpha < \omega_1 \mid \exists p \in F\ p \Vdash^W_Q \text{“}\check{\alpha} \in \dot{S}\text{”}\}$ is stationary.

Proof. This is a usual construction by Laver's diamond sequence

$$f : \kappa \longrightarrow H_\kappa.$$

We construct P_α and $\dot{Q}_\alpha$ by recursion on α. Suppose we have constructed P_α such that $P_\alpha \in H_\kappa$. Let $\dot{Q}_\alpha$ be a P_α-name such that

- $1 \Vdash_{P_\alpha}$ “$\dot{Q}_\alpha$ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper”,
- If $f(\alpha)$ is a P_α-name, then $1 \Vdash_{P_\alpha}$“ if $f(\alpha)$ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, then $\dot{Q}_\alpha = f(\alpha)$”.

By the iteration lemma for semiproper + $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, for all $\alpha \leq \kappa$, P_α are semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper. In particular, $\langle C_\delta \mid \delta \in A\rangle$ remains tail club guessing in V^{P_κ}.

Claim 1. *The $+$-type forcing axiom holds for the preorders which are semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper in $W = V^{P_\kappa}$.*

Proof. Let G_κ be any P_κ-generic over V. We want to show the forcing axiom holds in $V[G_\kappa]$. Suppose $p \in G_\kappa$ and $p \Vdash_{P_\kappa}$“$\dot{Q}$ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper” and $p \Vdash_{P_\kappa}$“$\langle \dot{D}_i \mid i < \omega_1\rangle$ dense subsets of $\dot{Q}$ and $\dot{S}$ is a $\dot{Q}$-name of a stationary subset of ω_1 in the extensions via $\dot{Q}$”.

Choose an elementary embedding

$$j : V \longrightarrow M$$

such that

$$j(f)(k) = \dot{Q}.$$

Then in M, denote

$$P^M_\alpha = j(\langle P_i \mid i \leq \kappa \rangle)_\alpha$$

for $\alpha \leq j(\kappa)$ and

$$\dot{Q}^M_\alpha = j(\langle \dot{Q}_i \mid i < \kappa \rangle)_\alpha$$

for $\alpha < j(\kappa)$. We have

If $j(f)(\kappa)$ is a P^M_κ-name in M, then $1 \Vdash^M_{P^M_\kappa}$ "if $j(f)(k)$ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper, then $\dot{Q}^M_\kappa = j(f)(k)$".

Claim 2. *In $V[G_\kappa]$ with $p \in G_\kappa$, we have if $V[G_\kappa] \models$ "$Q = \dot{Q}_{G_\kappa}$ is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper", then $M[G_\kappa] \models$ "Q is semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper" .*

Proof. Let λ be a sufficiently large regular cardinal. We may assume

$$V \cap {}^\lambda M \subseteq M.$$

Therefore $P_\kappa = P^M_\kappa$. We also have

$$V[G_\kappa] \cap {}^\lambda M[G_\kappa] \subseteq M[G_\kappa],$$

because P_κ is a set.

So we are in a situation where we have the same objects to be preserved at the same critical level (see 13.4 theorem) with respect to Q from $V[G_\kappa]$ to $V[G_\kappa][G_Q]$ and from $M[G_\kappa]$ to $M[G_\kappa][G_Q]$ for the same Q-generic filters G_Q. Since $M[G_\kappa][G_Q] \subseteq V[G_\kappa][G_Q]$, we would be done.

Here are some details to show $M[G_\kappa] \models$ "Q is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper". We first observe the critical levels for Q in $V[G_\kappa]$ and in $M[G_\kappa]$ are the same. In $V[G_\kappa]$, we calculate

$$K^{V[G_\kappa]} = H^{V[G_\kappa]}_{(|\mathrm{TC}(Q)|^+)^{V[G_\kappa]}}$$

and in $M[G_\kappa]$

$$K^{M[G_\kappa]} = H^{M[G_\kappa]}_{(|\mathrm{TC}(Q)|^+)^{M[G_\kappa]}}.$$

We may assume they are common to $V[G_\kappa]$ and $M[G_\kappa]$. Let us denote $K = K^{V[G_\kappa]} = K^{M[G_\kappa]}$.
Also we may assume that

$$(Seq^\omega(K))^{V[G_\kappa]} = (Seq^\omega(K))^{M[G_\kappa]}$$

and $V[G_\kappa]$ and $M[G_\kappa]$ have the same ω-stationary sets in the common $Seq^\omega(K)$.
Namely, for all $S \subseteq (Seq^\omega(K))^{V[G_\kappa]} = (Seq^\omega(K))^{M[G_\kappa]}$ with $S \in V[G_\kappa]$ (iff $S \in M[G_\kappa]$),

S is ω-stationary in $V[G_\kappa]$ iff S is ω-stationary in $M[G_\kappa]$,

$S \cap (\mathcal{E} \uparrow K)$ is ω-stationary in $V[G_\kappa]$ iff $S \cap (\mathcal{E} \uparrow K)$ is ω-stationary in $M[G_\kappa]$.

$S \cap (\mathcal{E} \uparrow K)$ is ω-semistationary in $V[G_\kappa]$ iff $S \cap (\mathcal{E} \uparrow K)$ is ω-semistationary in $M[G_\kappa]$.

So if $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semistationary S remained in $V[G_\kappa][G_Q]$, then so in $M[G_\kappa][G_Q]$, where G_Q is Q-generic over $V[G_\kappa]$ (iff over $M[G_\kappa]$). Hence $M[G_\kappa] \models$ "Q is $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper".

We may similary show the semistationary sets at the critical level with respect to Q are all preserved from $M[G_\kappa]$ to the extensions $M[G_\kappa]^Q$. Hence $M[G_\kappa] \models$ "Q is semiproper".

□

Since $p \in G_\kappa$, we have in $M[G_\kappa]$

$$(\dot{Q}^M_\kappa)_{G_\kappa} = Q.$$

Extend $j : V \longrightarrow M$ to $j : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}]$, where $G_{\kappa j(\kappa)}$ is $P^M_{\kappa j(\kappa)}$-generic over $V[G_\kappa]$. Note every condition in P_κ is equivalent to a condition with bounded support in P_κ. Hence we have

$$j(q) \equiv q^\frown 1 \in P^M_{j(\kappa)}$$

for $q \in G_\kappa$.

In $M[G_{\kappa+1}]$, we have a filter $F \subseteq Q$ such that for all $i < \omega_1$, $D_i \cap F \neq \emptyset$ and $\{\alpha < \omega_1 \mid \exists q \in F\ q \Vdash^{M[G_\kappa]}_Q \text{“}\alpha \in \dot{S}\text{”}\}$ is stationary.

We may assume Q is an ordinal so that for the restriction of j, we have

$$j\lceil Q \in {}^{<\lambda}M \cap V \subset M.$$

Then we have $M[G_{j(\kappa)}] \models$ “ $j\text{"}F \subseteq j(Q)$ is directed in $j(Q)$, for all $i < \omega_1$, $j(\langle D_l \mid l < \omega_1\rangle)_i \cap j\text{"}F \neq \emptyset$ and $\{\alpha \mid \exists q \in j\text{"}F\ \ q \Vdash^{M[G_{j(\kappa)}]}_{j(Q)} \text{“}\alpha \in j(\dot{S})\text{”}\ \}$ is stationary”.

Hence $V[G_\kappa] \models$ “$\exists F \subseteq Q$ directed, for all $i < \omega_1$ $D_i \cap F \neq \emptyset$ and $\{\alpha < \omega_1 \mid \exists q \in F\ q \Vdash^{V[G_\kappa]}_Q \text{“}\alpha \in \dot{S}\text{”}\}$ is stationary”

□

References

[B] M. Bekkali, *Topics in Set Theory*, Lecture Notes in Mathematics, vol. 1476 (1991), Springer-Verlag.

[F] Q. Feng, On Stationary Reflection Principles, *Proceedings of the 6th Asian Logic Conference*, World Scientific (1998) pp. 83-106.

[I] T. Ishiu, The saturation of club guessing ideals, Preprint, May 16, 2005.

[M] T. Miyamoto, A Limit Stage Construction for Iterating Semiproper Preorders, *Proceedings of the 7th and 8th Asian Logic Conferences*, World Scientific (2003) pp. 303-327.

[S] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Springer, 1998.